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Fracpairs and Quantitative Mereology

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Abstract

Fracpairs may be taken for a basic entity in arithmetic. A fracpair is the input pair for a division. A fracpair may also be called a ratio. Fracpairs can be built on any arithmetic, starting with naturals or integers. Fracpairs are taken as elements of an additional sort, for which additional operations may be introduced in a stepwise manner. These operations are mainly implementations of well-known from Elementary Arithmetic with Division (EAwD), for which however a ramification of different implementations arises due to the low level of abstraction of inputs. A fundamental interpretation of a fracpair is that it selects, or describes a selection, a number of parts of an entity which has been split in a number of comparable parts. The paper proposes various options for formalizing the split and select interpretations of fracpairs.

1 Introduction

The fraction definition problem was stated in some detail in Bergstra 2020 [5]. That paper fails to identify or produce a satisfactory definition of fractions and instead has a focus on explaining why giving such definitions is hard. In particular the suggestion is made that new terminology may be needed and notably "fracterm" is proposed to be suitable for obtaining more precise language about Elementary Arithmetic with Division (EAwD). In Bergstra & Tucker 2026 [20] besides fracterm (a term with division as its leading function symbol) additional terminology is used: fracvalue (the value of a closed fracterm), fracpair, fracsign, fractoken (fracsign occurrence), and fracton (also fraxion) is proposed as an ambiguous category ranging over: fracvalue, fracpair, fracterm, fracsign and fractoken (these concepts being listed in decreasing level of abstraction). An unambiguous concept which is clearly related to fraction, as used in EAwD, is called a landmark for it. Taking fracterms seriously has led to a family of fracterm calculi with the fracterm calculus for Suppes-Ono arithmetic (i.e. adopting $x/0 = 0$ for all x) as a primary example (see e.g. Suppes 1957 [28], Ono 1983 [27], Bergstra & Tucker 2007 [15], Bergstra, Bethke & Ponse 2013 [?] and 2015 [11], Bergstra & Middelburg 2016 [12], and Anderson & Bergstra 2021 [3]), and the fracterm calculus for common meadows (adopting $x/0 = \perp$) as a secondary example (see e.g. Bergstra & Tucker 2023 [18] and 2024 [19], Dias & Dinis 2023 [24] and 2024 [25]).

Fracpairs have been studied in Bergstra & Ponse 2016 [13] (where a general approach is taken which in particular provides an initial algebra for the equational axioms of common meadows) and Bergstra & Ponse 2020 [14] (where the question is discussed to what extent one may equip structures for EAwD with numerators and denominators as additional features for numbers, leading to a range of disparate observations). Below we will focus on fracpairs and take these for quantities which are entitled to their own sort with dedicated operations for that sort. We will not take the general approach of [13] and instead we will assume that fracpairs are pairs of values which are digital integers (DNN-integers, for decimal natural number integers) as discussed in Bergstra 2020 [6]. DNN-naturals are the signs $0, 1, 2, ..., 10, 11, 12, ..$ and DNN-integers in addition contain $-p$ for p a non-zero DNN). By speaking of DNN-integers we comply with the reservations about defining numbers from Benacerraf 1965 [4]. Thus with naturals and integers below, we refer to DNN-naturals and DNN-integers. We may in some cases include $\perp$ as a peripheral element in the sort of integers, thus arriving at $\mathrm{DNN}_{\perp}$-integers.

1.1 Material metaphors

In explanations of EAwD much use is made of explanation by way of (stories about) splitting and reconfiguring material objects like cakes and pies. We will makes use of material metaphors as well and in particular we make use of the idea that line segments l_1, l_2 etc. can be split into parts of equal sizes, and that groups of parts can be reconfigured to new objects, which we will call configurations. The parallel composition operator $-||_{\mathsf{d}}-$ is used to combine configurations, by aligning these in direction d along a hypothetical infinite line: $(X||_{\mathsf{d}}Y)||_{\mathsf{d}}(Z||_{\mathsf{d}}U)$ denotes a configuration which, by way of bracketing still preserves an account of its construction history. We will use fracpairs to measure the size (various forms of length) of configurations. Fracpair calculus provides a fairly clear account of the one-dimensional situation. We will discuss fracpairs in the setting of Suppes-Ono arithmetic and in the setting of common meadows. Working with other arithmetical models such as transrationals or wheels requires straightforward modifications which we will not spell out in this paper.

1.2 Fracpairs

In connection with fracpairs we will provide additional notation and terminology: $\frac{a}{b}\!\bullet$ for the fracpair with a as its numerator and b as its denominator. It is assumed when using $\frac{a}{b}\!\bullet$ that fracpairs are found in an additional sort FP of fracpairs. It will be assumed that a and b are values taken from a relevant domain for arithmetic, e.g. a ring or a field or a common meadow or transrational arithmetic, a wheel, an entropic transreal arithmetic or any other arithmetic which one intends to consider.

We find that fracpairs may be simple, when the numerator and denominator have integer values, or composed otherwise, and a simple fracpair may or may not be simplified (also termed reduced). Further a unit fracpair has 1 as its denominator an in case of an ordered arithmetic a proper fracpair has $1 < a < b$. A fracpair is safe if $b \neq 0$ (and in the presence of $\perp$ if also $b \neq \perp$).

Some new terminology may be helpful: in an arithmetic with a sort of fracpairs one assumes the presence of arithmetical operators, addition, multiplication, subtraction, opposite (additive inverse), division and (multiplicative) inverse of taking arguments in the sort FP and producing results in that sort as well. A complication is that operators may have different definitions, which indeed produce different results for fracpairs. Using subscripts such matters may be clarified. For instance $- +_{\mathsf{A}} -$ is the familiar addition operator defined by

$$\frac{x}{y}\!\bullet +_{\mathsf{A}} \frac{u}{v}\!\bullet = \frac{x \cdot v + y \cdot u}{y \cdot u}\!\bullet$$

With the help of a conditional operator another operator for addition $-+_{\mathsf{B}}$ $-$ may be defined which guarantees for the addition of simple fracpairs that

$$\frac{a}{c}\!\bullet +_{\mathsf{A}} \frac{b}{c}\!\bullet \equiv_{\mathsf{bs}} \frac{a+b}{c}\!\bullet$$

where $\equiv_{\mathsf{bs}}$ is an equivalence on fracpairs which differs from equality only in the presence of $\perp$ as a value. Yet another operator for addition $- +_{\mathsf{C}} -$ may be defined which guarantees for the addition of simple fracpairs that $\frac{a}{b}\!\bullet +_{\mathsf{C}} \frac{c}{d}\!\bullet = \frac{e}{f}\!\bullet$, with f the smallest common multiple by a non-negative factor of b and c.

Multiplication can be done in different ways as well, we will work throughout the paper with:

$$\frac{a}{b}\!\bullet \cdot_{\mathsf{A}} \frac{c}{d}\!\bullet = \frac{a \cdot c}{b \cdot d}\!\bullet$$

For division we will assume that division, which may also appear in different forms (say $\frac{-}{-}\star$) is defined using inverse, which may also take different forms $(-)^{-1_\star}$:

$$\frac{\frac{x}{y}\bullet}{\frac{u}{v}\bullet}\star = \frac{x}{y}\bullet \cdot (\frac{u}{v}\bullet)^{-1_\star}$$

and we will merely use inverse, while taking care of its different versions by way of subscripting. We have as the main version of inverse $(-)^{-1_\mathsf{A}}$ given by

$$(\frac{x}{y}\bullet)^{-1_\mathsf{A}} = \frac{y \cdot y}{x \cdot y}\bullet$$

The definition of $(-)^{-1_\mathsf{A}}$ reflects inverse in a common meadow. We will have no variation for the opposite of a fracpair: $-\frac{a}{b}\bullet = \frac{-a}{b}\bullet$, and subtraction varies together with addition and is always given as

$$\frac{x}{y}\bullet -_\star \frac{u}{v}\bullet = \frac{x}{y}\bullet +_\star (-\frac{u}{v}\bullet)$$

1.3 Fracpterms and fracpterm flattening

An expression of sort FP is a fracpterm. Fracpterms need not have fracpairing as the outermost function symbol, an example is:

$$(-\frac{5}{2 \cdot x}\bullet) +_\mathsf{A} (\frac{y-1}{z \cdot x + 1}\bullet \cdot_\mathsf{A} \frac{1}{y-2}\bullet)^{-1}$$

A fracpterm is flat if its outermost function symbol is fracpairing. The operator definitions as given above allow fracpterm flattening: every fracpterm is provably equal to a flat fracpterm. The proof is immediate by induction on the complexity of fracpterms.

If one starts out with a field of rationals a sort FP may be added based on the Suupes-Ono convention that division by zero produces zero. Now we find that frapcterm flattening fails, in particular there is no way to add fracpterms in such a manner that the result is a flat fracpterm. Following Bergstra, Bethke & Ponse [10] all fracterms may be proven equal to a sum of flat fracterms, and according to Bergstra & Middelburg 2016 [12] sums of arbitrary size may be needed. It follows that in a fracpterm calculus style Suppes-Ono arithmetic over a field of rationals (or in fact any field) every fracpterm can be proven equal to a substitution instance of $\frac{x_1}{y_1}\bullet + \ldots + \frac{x_n}{y_n}\bullet$ for some natural $n > 0$. As mentioned $\frac{x_1}{y_1}\bullet + \ldots + \frac{x_{n+1}}{y_{n+1}}\bullet$ cannot be rewritten as a substitution instance of $\frac{x_1}{y_1}\bullet + \ldots + \frac{x_n}{y_n}\bullet$.

We notice that [?] demonstrates (in Suppes-Ono arithmetic), for all natural $n > 0$ the existence of a natural m and polynomials $p_1, \ldots, p_m$ and $q_1, \ldots, q_m$ each with variables among $x_1, \ldots, x_n, y_1, \ldots, y_1$ such that

$$(\frac{x_1}{y_1}\bullet + \ldots + \frac{x_n}{y_n}\bullet)^{-1} = \frac{p_1}{q_1}\bullet + \ldots + \frac{p_m}{q_m}\bullet$$

We were unable to find attractive forms for these polynomials, however.

1.4 Working with fracpairs

Intuitive explanations of a fracpair, as well as examples of the usage of fracpairs, say of $\frac{2}{7}\bullet$, may be given in different forms. We will discuss various usages of fracterms.

1.4.1 Input pair for division

The pair of both arguments for a division, an explanation for fracpairs which makes sense only once the concept of a rational number has been adopted, or in case integer division (either as a partial function or as a total function with remainder) is discussed first as a preliminary step. For fracpairs as inputs for division we have several notions which are derived from conventional arithmetic. The $\frac{x}{y}\bullet$ allows the following qualifications:

(i) simple fracpair: in case x and y are integer values,

(ii) safe fracpair: a simple fracpair with y non-zero.

(iii) simplified fracpair: simple fracpair with $\gcd(x, y) = 1$ if $x \cdot y \neq 0$ and $x = 1$ if $y = 0$, and $y = 1$ if $x = 0$. Thus $\frac{2}{4}\bullet$ is not simplified while $\frac{1}{2}\bullet$ and $\frac{3}{4}\bullet$ are simplified, so that besides $\frac{2}{4}\bullet \neq \frac{3}{4}\bullet$ also $\frac{2}{4}\bullet \neq \frac{1}{2}\bullet$.

(iv) unit fracpair: in case $x = 1$,

(v) proper fracpair: simple fracpair with $0 \leq x$ and $0 < y \leq x$

(vi) composite fracpair: not a simple fracpair.

(vii) peripheral fracpair: either one or both of x and y are peripheral values (e.g. $\perp$, ∞, $-\infty$, Φ).

1.4.2 Fracpairs for splitting and grouping

The focus of the paper will be on other uses of fracpairs, which usages differ from mere pairing of inputs for a division, in particular forms of use where $\frac{a}{b}\bullet$ serves as a value which indicates that in some way a parts are to be taken or selected upon the splitting of an object in b equal or comparable parts. An operator $\circ$ which may be decorated with various subscripts and superscripts will be used to present the application of $\frac{a}{b}\bullet$ in such ways. The work has a focus on elementary composition and decomposition of objects, a theme which belongs to mereology, and due to a focus on counting we will speak of quantitative mereology.

1.4.3 Fracpairs as a carrier of elementary arithmetic with fracpairing (EAwFP)

A simplified version of arithmetic does away with division and takes a focus on fracpairs as the major concept regarding the division at hand. The idea is that an account of EAwFP can be given without any mention of division of numbers (fracvalues) serving as quotients. We hold that fracpairs are bonafide quantities which allow for practical application in quantitative mereology.

2 Sub-collections of partitioned objects

For this intuition one only admits various forms of addition on fracpairs. One imagines an object (typically a cake or a pie) which is split in a number, say k, of parts with the same shape and surface. Then one imagines a collection of n of these parts, first of all with $n \leq k$. Now one takes an interest in "how much of the object has been collected". First of all with $\frac{n}{k}\bullet$ we denote any collection as specified above. Here one may notify a substantial abstraction because no attention is paid to precisely which parts have been selected. Thus we understand, say $\frac{3}{7}\bullet$, as a qualification of the selection made: it may be understood as an attribute of the selection, and different selections may have the same attribute. We may list the parts of a whole by names: a, b, c, d, e, f, g, which are taken for parts of equal size and form of an object H. Now a selection may be given as: $\mathsf{selection}\{a, d, e\}\mathsf{from\_partition}_H\{a, b, c, d, e, f, g\}$. Now we measure the selection by way of a fraicpair

$$\mathsf{fracpair}(\mathsf{selection}\{a, d, e\}\mathsf{from\_partition}_H\{a, b, c, d, e, f, g\}) = \frac{3}{7}\bullet$$

and

$$\mathsf{fracpair}(\mathsf{selection}\{a, d, e, g\}\mathsf{from\_partition}_H\{a, b, c, d, e, f, g\}) = \frac{4}{7}\bullet$$

We may introduce abbreviations "sel" for selection, "par" for partition, and "f" for "from", thus obtaining, say

$$\mathsf{fracpair}(\mathsf{sel}\{a, b\}\mathsf{f\_par}_H\{a, b, c, d, e\}) = \frac{2}{5}\bullet$$

We introduce $\equiv_{\mathsf{volume}}$ which expresses that two selections from partitions of H when each of these is taken together amount to the same volume of the same stuff ("the stuff of H"). We find:

$$\mathsf{sel}\{a, b\}\mathsf{f\_par}_G\{a, b, c, d\} \equiv_{\mathsf{volume}} \mathsf{sel}\{c, d\}\mathsf{f\_par}_G\{a, b, c, d\}$$
$$\mathsf{sel}\{a\}\mathsf{f\_par}_G\{a, b\} \not\equiv_{\mathsf{volume}} \mathsf{sel}\{a, b\}\mathsf{f\_par}_G\{a, b, c, d\}$$

and also

$$\mathsf{sel}\{a\}\mathsf{f\_par}_H\{a, b, c, d\} \not\equiv_{\mathsf{volume}} \mathsf{sel}\{b, c\}\mathsf{f\_par}_H\{a, b, c, d\}$$

Assuming that G and H are different stuff:

$$\mathsf{sel}\{a\}\mathsf{f\_par}_G\{a, b\} \not\equiv_{\mathsf{volume}} \mathsf{sel}\{a\}\mathsf{f\_par}_H\{a, b\}$$

2.1 Fracpairs as determinants for selections

We may strengthen the notation by writing $\frac{2}{5}\bullet \circ H$, which provides the fracpair with an entity which is to be split in 5 ordered parts of equal size and form, and from which the first two are selected. The fracpair split-select notation combines with volume equivalence as follows:

$$\frac{1}{2}\bullet \circ G \equiv_{\mathsf{volume}} \frac{2}{4}\bullet \circ G$$

With $n \circ G$ we denote a composite made out of n copies of G. Now for compositions of objects several equivalences may be distinguished:

(i) $\equiv_{\mathsf{volume}}$ an equivalence which may be generalized to compositions of different kinds of matter each of which are then supposed to be present in equal volumes. We find

$$2 \circ \frac{1}{3}\bullet \circ G \equiv_{\mathsf{volume}} \frac{2}{3}\bullet \circ G$$

(ii) $\equiv_{\mathsf{volume/origin}}$ an equivalence which is more demanding than $\equiv_{\mathsf{volume}}$ by requiring in addition that the equal volumes are mere reconfigurations of the same substance. We find

$$2 \circ \frac{1}{3}\bullet \circ G \not\equiv_{\mathsf{volume/origin}} \frac{2}{3}\bullet \circ G$$

(iii) $\equiv_{\mathsf{volume/origin/split}}$, a yet more demanding equivalence which requires that a reconfiguration leaves all parts unaffected. For instance:

$$\frac{1}{2}\bullet \circ G \not\equiv_{\mathsf{volume/origin/split}} \frac{2}{4}\bullet\circ,\ \frac{1}{2}\bullet \circ G \equiv_{\mathsf{volume/origin}} \frac{2}{4}\bullet \circ G$$

2.2 Subcollections of positively partitioned objects

In quantitative mereology we perform elementary calculations to objects and parts thereof. For instance with $\frac{2}{7}\bullet$ we may "quantify" the condition which features 2 parts from a divided entity which has been split in 7 parts of equal size (or weight, or surface, whatever metric is suitable for the kind of entity at hand). In contrast $\frac{-2}{7}\bullet$ denotes a debt of size two of an entity which has been split in 7 parts (i.e. two more such parts are due). The selection is given by a marking (or color or whatever feature appropriate for the kind of entity at hand) for each of the selected parts, while debt is given a second marking. The opposite of debt is "ordinary volume": $-\frac{-2}{7}\bullet = \frac{2}{7}\bullet$. When one looks at subcollections of equipartitioned objects one finds that an interpretation of $\frac{2}{-3}\bullet$ seems to require the intuition of an object which has been spit in a negative number of parts (in this case -3). We wish to prevent an incentive for such counterintuitive explanations, by rewriting fracpairs with a negative denominator into one's with a positive denominator. The situation may be understood as working with fracpairs though adopting the following equivalence relation (with "sse" for sign swapping equivalence) among those:

$$\frac{-x}{y}\bullet \equiv_{\mathsf{sse}} \frac{x}{-y}\bullet$$

We notice that in Anderson 2025 [1] sign swapping is discussed (p28) under the name rectification. By adopting the above rule one removes "negative splitting" of objects, and instead of $\frac{2}{-8}\bullet$ one will merely contemplate $\frac{-2}{8}\bullet$ which may be taken for being in debt two parts of a split into 8 parts (of a stuff which has been left implicit). The rule provides a closer intuition between the world of signs and the world which is modeled by means of these signs.

2.3 Differences between fracpairs

The distinction between $\frac{1}{3}\bullet$ and $\frac{2}{6}\bullet$ is harder to illustrate than its equivalence. But we may think in terms of H as a cake and the we find that obtaining $\frac{1}{3}\bullet \circ H$ may be useful (as it can be consumed) while obtaining $\frac{500}{1500}\bullet \circ H$ may be a useless mess which resist proper consumption. Perhaps a complex process of reconstruction may turn $\frac{500}{1500}\bullet \circ H$ into $\frac{10}{30}\bullet \circ H$ which then may allow proper consumption. We find that differences between $\frac{n}{3 \cdot n}\bullet \circ H$ for various natural numbers n above 0 are a gradual matter.

Bread is often sold in sliced form, say:

$$\frac{25}{25}\bullet \circ [\mathsf{bread}] \neq \frac{2}{2}\bullet \circ [\mathsf{bread}]$$

expressing that sliced bread is not the same as two unsliced parts of the same bread. Instead of addition we apply parallel composition (ordered juxtaposition) to mereological compositions:

$$\frac{10}{25}\bullet \circ [\mathsf{bread}] || \frac{5}{25}\bullet \circ [\mathsf{bread}] = \frac{15}{25}\bullet \circ [\mathsf{bread}]$$

whereas

$$\frac{5}{10}\bullet \circ [\mathsf{bread}] || \frac{10}{20}\bullet \circ [\mathsf{bread}] \neq \frac{20}{20}\bullet \circ [\mathsf{bread}]$$

although

$$\frac{5}{10}\bullet \circ [\mathsf{bread}] || \frac{10}{20}\bullet \circ [\mathsf{bread}] \equiv_{\mathsf{volume}} \frac{20}{20}\bullet \circ [\mathsf{bread}]$$

However, more often than not slices of bread have different sizes, and this way of modeling the matter may lack sufficient precision for practical application.

$$\frac{n}{m}\bullet \circ (H||K) = (\frac{n}{m}\bullet \circ H)||(\frac{n}{m}\bullet \circ (H||K)$$

2.4 Notational complications

One may ask whether or not

$$\frac{5}{10}\bullet \circ [\mathsf{bread}] || \frac{10}{20}\bullet \circ [\mathsf{bread}] \equiv_{\mathsf{volume/origin}} \frac{20}{20}\bullet \circ [\mathsf{bread}]$$

Now several complications arise: are both occurrences of bread in the LHS supposed to denote the same bread, and if so, is the second occurrence the residue after splitting and selecting according to $\frac{5}{10}\bullet\circ$ and if so, is every slice that has not been selected split into 20 parts in the next step? More generally, are we dealing with ordinary algebra or with operations with possible side effects on various components of a composition. Finding precise answers on such questions requires refinement of the notation so that different scenarios may be made explicit.

2.5 More on cake and pie dimensions

With $\frac{2}{7}\bullet \circ [\mathsf{cake}\ 10]$ we denote a cake of size 10 which has been split in 7 parts of which 2 parts have been marked, and with $\frac{3}{5}\bullet \circ [\mathsf{pie}\ 15]$ we denote a pie of size 15 which has been split in 5 parts of which 3 parts have been marked. Now we notice that composition of selected fragments of cakes (or pies) makes sense. We may write

$\frac{2}{7}\bullet \circ [\mathsf{cake}\ 10] = \frac{2}{14}\bullet \circ [\mathsf{cake}\ 20] = \frac{2}{7}\bullet \circ [\frac{1}{2}\bullet \circ [\mathsf{cake}\ 20]] = (\frac{2}{7}\bullet \cdot_{\star} \frac{1}{2}\bullet) \circ [\mathsf{cake}\ 20]$

We may write $(\frac{x}{y}\bullet || \frac{u}{v}\bullet)\circ$ for $\lambda H.(\frac{x}{y}\bullet \circ H)||(\frac{u}{v}\bullet) \circ H)$ so that $\frac{2}{7}\bullet \circ [\mathsf{cake}\ 10]||\frac{2}{5}\bullet \circ [\mathsf{cake}\ 10] = (\frac{2}{7}\bullet || \frac{3}{5}\bullet) \circ [\mathsf{cake}\ 10]$. However, we notice that $(\frac{x}{y}\bullet || \frac{u}{v}\bullet)$ is not itself a fracpair and that $-||-$, unlike addition and multiplication, cannot be understood as an operator on fracpairs. Some plausible equations may be found:

$$\frac{2}{7}\bullet \circ [\mathsf{cake}\ 10]||\frac{3}{5}\bullet \circ [\mathsf{cake}\ 20] = (\frac{4}{14}\bullet || \frac{3}{10}\bullet) \circ [\mathsf{cake}\ 20]$$

Composition of pies and cakes may be forbidden (prevented). $\frac{2}{7}\bullet \circ [\mathsf{cake}\ 10]||\frac{3}{5}\bullet \circ [\mathsf{pie}\ 9] = \frac{2}{7}\bullet \circ [\mathsf{cake}\ 10]||\frac{3}{5}\bullet \circ [\mathsf{pie}\ 10] = \perp$

2.6 Restrictions on composition

we may be even more strict on addition and equality of cakes and pies, under the assumption that comparison and addition only matters for cakes (or pies), that is for cakes/pies with the same size. One finds for instance:

$\frac{2}{3}\bullet \circ [\mathsf{cake}\ 15] = \frac{1}{2}\bullet \circ [\mathsf{cake}\ 20]$
$\frac{3}{7}\bullet \circ [\mathsf{cake}\ 13] \neq \frac{3}{14}\bullet \circ [\mathsf{cake}\ 25]$
$\frac{2}{7}\bullet \circ [\frac{1}{2}\bullet \circ [\mathsf{cake}\ 10]] = (\frac{2}{7}\bullet \cdot \frac{1}{2}\bullet)[\mathsf{cake}\ 10]$
$\frac{2}{7}\bullet \circ [\mathsf{cake}\ 10]||\frac{3}{5}\bullet \circ [\mathsf{cake}\ 15] = \perp$
$\frac{2}{7}\bullet \circ [\mathsf{cake}\ 8]||\frac{2}{7}\bullet \circ [\mathsf{pie}\ 8] = \perp$

2.7 Further restrictions on composition

We may be yet more strict on addition and equality of cakes and pies, under the assumption that comparison and addition only matters for the same cakes (or pies), that is in any case only for cakes/pies with the same size. One finds for instance:

$\frac{2}{3}\bullet \circ [\mathsf{cake}\ 15] \neq \frac{1}{2}\bullet \circ [\mathsf{cake}\ 20]$
$\frac{3}{7}\bullet \circ [\mathsf{cake}\ 15] \neq \frac{3}{14}\bullet \circ [\mathsf{cake}\ 15]$ The above experiments indicate that rules of calculation are a matter of design, intimately connected with what it is that one intends to model.

2.8 Disjoint union of collections

Another matter which comes into play is taking disjoint unions of selections of partitions of objects of the same stuff. The notation will be more complicated, such as:

$$(\mathsf{sel}\{a\}\mathsf{f\_par}_G\{a,b\}) \cup_{\mathsf{d}} (\mathsf{sel}\{b\}\mathsf{f\_par}_G\{a,b\}) = \mathsf{sel}\{a,b\}\mathsf{f\_par}_G\{a,b\}$$

Addition of fracpairs (in various forms) will play a role in analyzing when such disjoint unions have the same volume.

Now we may introduce disjoint union addition of fracpairs with these rules:

$$\frac{x}{z}\bullet +_{\mathsf{du}} \frac{y}{z}\bullet = \frac{x+y}{z}\bullet$$

In all other cases an error is found, and such may be expressed by way of different rules: for instance with X, Y ranging over fracpairs:

$$\mathsf{Denom}(X) = \mathsf{Denom}(Y) \rightarrow X +_{\mathsf{du}} Y = \frac{\mathsf{Num}(X) + \mathsf{Num}(Y)}{\mathsf{Denom}(X)}\bullet$$

$$\mathsf{Denom}(X) \neq \mathsf{Denom}(Y) \rightarrow X +_{\mathsf{du}} Y = \frac{\perp}{\perp}\bullet$$

With variables for numbers only one finds for instance:

$$x \neq u \rightarrow \frac{x}{z}\bullet +_{\mathsf{du}} \frac{y}{u}\bullet = \frac{\perp}{\perp}\bullet$$

$$\frac{x}{y}\bullet +_{\mathsf{du}} \frac{\perp}{\perp}\bullet = \frac{\perp}{\perp}\bullet +_{\mathsf{du}} \frac{x}{y}\bullet = \frac{\perp}{\perp}\bullet$$

3 Cutting a sliced cake in two dimensions

With $- \circ_{\mathsf{d}} -$ we may denote that two successive acts of slicing (cuts) are applied in perpendicular direction. Thus, with

$$\frac{4}{4}\bullet \circ_{\mathsf{d1}} \frac{10}{18}\bullet \circ_{\mathsf{d2}} [\mathsf{cake}\ 25]$$

we denote that a sliced cake (10 slices taken out of 18 slices making up the whole cake) the cake having size (here length 25), is cut in four parts in a perpendicular manner such that 40 parts of equal size result. $\frac{2}{4}\bullet \circ_{\mathsf{d1}} \frac{10}{18}\bullet \circ_{\mathsf{d2}} [\mathsf{cake}\ 25]$ selects 20 of these slices.

The function $\#_{\mathsf{parts}}(-)$ yields the number of parts in a collection. We find $\#_{\mathsf{parts}}(\frac{3}{4}\bullet \circ_{\mathsf{d1}} \frac{7}{18}\bullet \circ_{\mathsf{d2}} [\mathsf{cake}\ 25]) = 3 \cdot 7 = 21$

3.1 Cutting a slice in three dimensions

One more prefixing (that is applying $\frac{n}{m}\bullet\circ$ describes the combination of $m-1$ cuts in equal slices followed by a selection of n of the slices. Thus

$$\#_{\mathsf{parts}}(\frac{2}{5}\bullet \circ_{\mathsf{d1}} \frac{3}{4}\bullet \circ_{\mathsf{d2}} \frac{7}{18}\bullet \circ_{\mathsf{d3}} [\mathsf{cake}\ 25]) = 2 \cdot 3 \cdot 7 = 42$$

Let the size (as given by weight) of cake 25 be 720 KG then

$$\mathsf{size}_{\mathsf{parts}}(\frac{2}{5}\bullet \circ_{\mathsf{d1}} \frac{3}{4}\bullet \circ_{\mathsf{d2}} \frac{7}{18}\bullet \circ_{\mathsf{d3}} [\mathsf{cake}\ 25]) = \frac{1}{5 \cdot 4 \cdot 18}\bullet \circ_{\mathsf{d:size}} [\mathsf{size\ cake}\ 25] =$$

$$\frac{1}{360}\bullet \circ_{\mathsf{d:size}} [720\ \mathsf{G}] = 2\ \mathsf{G}$$

The total weight of the selection is obtained by multiplying the weight per part, i.e. 2G and the number of parts, i.e. 42, that is 82G.

3.2 Further notational options

Instead of adopting the convention that successive cuts are made in different dimensions (length, breadth, depth) one may use subscripts for the decomposition function $\circ$ for instance:

$$\#_{\mathsf{parts}}(\frac{2}{5}\!\bullet \circ_{\mathsf{depth}} \frac{3}{4}\!\bullet \circ_{\mathsf{breadth}} \frac{7}{18}\!\bullet \circ_{\mathsf{length}} [\mathsf{cake}\ 25]) = 42$$

Here it matters that brackets are left implicit but ought to be written as follows:

$$\#_{\mathsf{parts}}(\frac{2}{5}\!\bullet \circ_{\mathsf{depth}} (\frac{3}{4}\!\bullet \circ_{\mathsf{breadth}} (\frac{7}{18}\!\bullet \circ_{\mathsf{length}} [\mathsf{cake}\ 25]))) = 42$$

which is another way of writing:

$$\#_{\mathsf{parts}}((\frac{2}{5}\!\bullet \circ_{\mathsf{depth}} (\frac{3}{4}\!\bullet \circ_{\mathsf{breadth}} (\frac{7}{18}\!\bullet \circ_{\mathsf{length}})))[\mathsf{cake}\ 25]) = 42$$

The latter notation conforms with the conventional usage of $\circ$ for function composition with $f \circ g$ representing f after g.

4 Equivalences on fracpairs

There is a range of equivalence relations on fracpairs which may be of use for fracpair calculus. In this section we will survey some options. Above in Paragraph 2.2 we have already seen sign swapping equivalence $\equiv_{\mathsf{sse}}$.

4.1 Bottom swapping

We may assume that $\perp$ constitutes an error element (named "bottom") in arithmetic with numbers, i.e. with naturals and integers. Then $x + \perp = x \cdot \perp = -\perp = \perp$. Now it is plausible not to make a significant distinction between fracpairs involving $\perp$, thereby finding $\equiv_{\mathsf{bse}}$ (for bottom swapping equivalence) given by:

$$\frac{x+\perp}{y}\!\bullet \equiv_{\mathsf{bse}} \frac{x}{y+\perp}\!\bullet$$

It the follows that

$$\frac{\perp}{y}\!\bullet \equiv_{\mathsf{bse}} \frac{\perp+\perp}{y}\!\bullet \equiv_{\mathsf{bse}} \frac{\perp+\perp}{y+\perp}\!\bullet \equiv_{\mathsf{bse}} \frac{\perp}{\perp}\!\bullet$$

and in the same way one finds

$$\frac{x}{\perp}\!\bullet \equiv_{\mathsf{bse}} \frac{\perp}{\perp}\!\bullet$$

As an application of bottom swapping we assume that the underlying arithmetic is a common meadow and consider the following definition of fracpair addition $+_{\mathsf{B}}$, following method B in the terminology of Anderson 2026 [2]. In the definition we use a conditional operator $- \lhd - \rhd -$ which satisfies: $x \lhd 0 \rhd y = y, x \lhd \perp \rhd y = \perp$ and $y \neq 0 \to x \lhd y \rhd z = x + 0 \cdot y$. Then the definition of $+_{\mathsf{B}}$ reads as follows:

$$\frac{x}{y}\bullet +_{\mathsf{B}} \frac{u}{v}\bullet = \frac{(x \cdot v + y \cdot u) \lhd y - v \rhd (x+u)}{(y \cdot v) \lhd y - v \rhd y}\bullet$$

Now one may take $v = y$ and expect to find find $\frac{x+v}{y}\bullet$ while in fact:

$$\frac{(x \cdot y + y \cdot u) \lhd y - y \rhd (x+u)}{(y \cdot y) \lhd y - y \rhd y}\bullet = \frac{x + u + 0 \cdot y}{y}\bullet$$

With $x = 1, u = 1, y = \bot$ instead of $\frac{1+1}{\bot}\bullet$ one finds $\frac{1+1+0\cdot\bot}{\bot}\bullet$ But $\frac{1+1}{\bot}\bullet = \frac{2}{\bot}\bullet \neq \frac{\bot}{\bot}\bullet = \frac{2+\bot}{\bot}\bullet = \frac{1+1+\bot}{\bot}\bullet = \frac{1+1+0\cdot\bot}{\bot}\bullet$. Working with $\equiv_{\mathsf{bse}}$, however one finds that

$$\frac{x}{y}\bullet +_{\mathsf{B}} \frac{u}{y}\bullet \equiv_{\mathsf{bse}} \frac{x+u}{y}\bullet$$

A second form of multiplication $- \cdot_{\mathsf{B}} -$ is defined as follows:

$$\frac{x}{y}\bullet \cdot_{\mathsf{B}} \frac{u}{v}\bullet = \frac{(((x \cdot u) \lhd x - v \rhd u) \lhd y - u \rhd (x \lhd x - v \rhd 1))) \lhd x \cdot v \rhd 1}{(((y \cdot v) \lhd x - v \rhd y) \lhd y - u \rhd (v \lhd x - v \rhd 1))) \lhd x \cdot v \rhd 0}\bullet$$

4.2 Initial common meadow equivalence

There are plenty of relevant equivalence relations on fracpairs. We mention an example which arose in original work on fracpairs. A useful equivalence on fracpairs is found by adopting

$$\frac{x}{y \cdot z}\bullet \equiv_{\mathsf{imcme}} \frac{x \cdot z}{y \cdot z \cdot z}\bullet$$

This equivalence on fracpairs has been studied in detail in Bergstra & Ponse [13]. The equivalences found in this manner are precisely those which follow from the equational axioms for common meadows (and "imcme" stands for initial model for common meadow equations). The equivalence $\equiv_{\mathsf{imcme}}$ is non-trivial as for instance

$$\frac{4}{4}\bullet = \frac{2 \cdot 2}{1 \cdot 2 \cdot 2}\bullet \equiv_{\mathsf{imcme}} \frac{2}{1 \cdot 2}\bullet = \frac{2}{2}\bullet \not\equiv_{\mathsf{imcme}} \frac{1}{1}\bullet$$

4.3 Quantity equivalence

We assume that in the underlying arithmetic (possibly without division) $x \cdot y = \bot \to x = \bot \vee y = \bot$. Now an equivalence $\equiv_{\mathsf{qe}}$ is defined as follows

$$\frac{x}{y}\bullet \equiv_{\mathsf{qe}} \frac{u}{v}\bullet$$

holds if at least one of the following three conditions is met:

(i) $y \neq 0 \wedge v \neq 0 \wedge y \cdot v \neq \bot$ and $x \cdot v = y \cdot u$,
(ii) $y = 0 \wedge v = 0 \wedge x \cdot u \neq \bot$, and
(iii) $x \cdot y = \bot$ and $u \cdot v = \bot$.

We find for instance:

$$\frac{1}{2}\bullet \equiv_{\mathsf{qe}} \frac{2}{4}\bullet, \frac{1}{0}\bullet \equiv_{\mathsf{qe}} \frac{2}{0}\bullet, \frac{1}{0}\bullet \not\equiv_{\mathsf{qe}} \frac{1}{\bot}\bullet \equiv_{\mathsf{qe}} \frac{2}{\bot}\bullet \equiv_{\mathsf{qe}} \frac{\bot}{2}\bullet \equiv_{\mathsf{qe}} \frac{\bot}{\bot}\bullet$$

and

$$x \equiv_{\mathsf{bse}} Y \to X \equiv_{\mathsf{qe}} Y$$

4.3.1 Quantify equivalence without $\perp$

In the absence of $\perp$ in the underlying arithmetic (for instance if a ring or field is given for that role, or an involutive meadow), the definition of quantity equivalence can be simplified as follows:

$$\frac{x}{y}\!\bullet \equiv_{\mathsf{qe}} \frac{u}{v}\!\bullet$$

holds if at least one of the following two conditions is met:
(i) $y = 0$ and $v = 0$
(ii) $y \neq 0$ and $v \neq 0$ and $y \cdot u = x \cdot v$.

4.3.2 Strong quantity equivalence

Strong quantity equivalence is implied by quantity equivalence (i.e. $\frac{x}{y}\!\bullet \equiv_{\mathsf{qe}} \frac{u}{v}\!\bullet \to \frac{x}{y}\!\bullet \equiv_{\mathsf{sqe}} \frac{u}{v}\!\bullet$) and also adopts:

$$\frac{x}{0}\!\bullet \equiv_{\mathsf{sqe}} \frac{\perp}{\perp}\!\bullet$$

5 Blocks

Given a direction/dimension d a block is an object which in the direction of d has not been split. The simplest idea of blocks is found if objects are extended in one direction only, that is material implementations of (finite segments of) lines and composition is putting such segments next to one-another. We will refer to these as linear blocks. A ground block is a block which has not been composed from other blocks. One may start with several ground blocks $l_1, l_2, ..$

Compositions of blocks (with $-||_{\mathsf{d}}-$ are again blocks. Blocks have a nested structure, and nesting is indicated with bracket pairs. For instance: $l_1||_{\mathsf{d}}l_2$, $((l_2||_{\mathsf{d}}l_3)||_{\mathsf{d}}l_4)||_{\mathsf{d}}l_2$ Starting with ground blocks one may construct larger blocks which may differ in structure while having the same size. Each block is a composition made up from ground blocks. Bracketing provides structure which does not contribute to size. However, we will identify $(l_2||_{\mathsf{d}}l_3)||_{\mathsf{d}}l_4$ and $((l_2||_{\mathsf{d}}l_3))||_{\mathsf{d}}l_4$. A more precise notion of blocks and compositions of blocks may be given along the lines of Bergstra 2024 [8].

For linear blocks in dimension d we have intuitions of length (or size). Composition of multi-blocks. As long as length can be measured with DNN-naturals the following equation is convincing for blocks X and Y:

$$\mathsf{length}_{\mathsf{DNN}}(X||_{\mathsf{d}}Y) = \mathsf{length}_{\mathsf{DNN}}(X) + \mathsf{length}_{\mathsf{DNN}}(Y)$$

Ground blocks l_a and l_b may have different lengths, say and only a fracpair

$$\mathsf{ratio}_{\mathsf{lengts}}(l_a, l_b) = \frac{\mathsf{length}(l_a)}{\mathsf{length}(l_b)}\!\bullet$$

may be available, which denotes the ratio between both lengths. It may now be useful to express lengths as fracpairs, starting with the case that $\mathsf{length}_{\mathsf{DNN}}(l_b)$ is known:

$$\mathsf{length}(l_b) = \frac{\mathsf{length}_{\mathsf{DNN}}(l_b)}{1}\!\bullet$$

now:

$$\mathsf{length}^{\mathsf{A}}(l_a) = \mathsf{length}(l_b) \cdot_{\mathsf{A}} \mathsf{ratio}_{\mathsf{lengts}}(l_a, l_b) = \frac{\mathsf{length}_{\mathsf{DNN}}(l_b)}{1}\!\bullet \cdot_{\mathsf{A}} \frac{\mathsf{length}(l_a)}{\mathsf{length}(l_b)}\!\bullet$$

5.1 Gluing

Subscripting the decomposition operator allows more detail in the specification of mereologic operations, here are some examples:

(i) Gluing parts together after splitting:

$$\frac{a}{b}\!\bullet \circ_{d,\mathsf{glue}/\mathsf{glue}} P$$

Split (by way of slicing) P in b equal parts along dimension d, and take the first a slices, glue these together, and then take the remaining slices and glue these together. The result is that P has been split in 2 parts.

(ii) Gluing selected parts together after splitting:

$$\frac{a}{b}\!\bullet \circ_{d,\mathsf{glue}/-} P$$

Split (by way of slicing) P in b equal parts along dimension d, and take the first a slices, glue these together. The result is that P has been split in $b - a + 1$ parts. The intuition is fairly clear with $a \leq b$, otherwise this interpretation is harder to make sense of.

(iii) Gluing parts together after splitting:

$$\frac{a}{b}\!\bullet \circ_{d,-/\mathsf{glue}} P$$

Split (by way of slicing) P in b equal parts along dimension d, and take the first a slices, and then take the remaining slices and glue these together. The result is that P has been split in $a + 1$ parts. The intuition is fairly clear with $a < b$, otherwise this interpretation is harder to make sense of.

(iv) Gluing only selected parts together after splitting:

$$\frac{a}{b}\!\bullet \circ_{d,\mathsf{glue}} P$$

5.2 Pseudo fracpairs

We will introduce some pseudo fracpairs, in particular $\frac{a}{\star}\!\bullet$ which indicates taking a parts (or fewer if no more are available) from an indeterminate number of parts of an entity which has been split into parts and has been put together in the order of dimension d. Take the first n parts of an object which has been split in several parts in dimension d:

$$\frac{a}{\star}\!\bullet \circ_d P$$

It follows that

$$\frac{a}{b}\!\bullet \circ_d P = \frac{a}{\star}\!\bullet \circ_d \frac{b}{b}\!\bullet \circ_d P$$

and if $a + b \leq c$

$$\frac{a}{b}\!\bullet \circ_d \frac{c}{c}\!\bullet \circ_d P = \frac{a}{\star}\!\bullet \circ_d \frac{a+b}{\star}\!\bullet \circ_d \frac{c}{c}\!\bullet \circ_d P$$

One may wish to take the second bundle of n parts of an object which has been split in several parts in dimension d:

$$\frac{a}{\star}\bullet \circ_d^2 P$$

The idea may be generalized to taking the n-th bundle:

$$\frac{a}{\star}\bullet \circ_d^n P$$

The new notation generalizes pseudo fracpair selection as follows:

$$\frac{a}{\star}\bullet \circ_d^1 P = \frac{a}{\star}\bullet \circ_d P$$

Now for a homogenous object composition P with all parts made out of the same stuff it is plausible that

$$\frac{2}{\star}\bullet \circ_d^1 \frac{7}{7}\bullet \circ_d P = \frac{2}{\star}\bullet \circ_d^3 \frac{7}{7}\bullet \circ_d P = \frac{2}{7}\bullet \circ_d P$$

5.2.1 Pseudo fracpairs applied to compositions

Pseudo fracpairs can be used in combination with composition of blocks.

$$P = (\frac{1}{\star}\bullet \circ_d^1 \frac{2}{2}\bullet \circ_d P)||_d(\frac{1}{\star}\bullet \circ_d^2 \frac{2}{2}\bullet \circ_d P) = \frac{2}{2}\bullet \circ_d P$$

Juxtaposition produces a sequence of objects, which may subsequently be glued to larger parts.

$\frac{\star}{3}\bullet \circ H$ takes H as a series of objects, for instance as produced by an application of $\frac{a}{b}\bullet\circ$, and then splits each of its components in 3 parts of similar shape and size. We find for instance:

$$\frac{\star}{3}\bullet \circ \frac{2}{5}\bullet \circ H = \frac{6}{15}\bullet \circ H$$

$$\frac{\star}{2}\bullet \circ ((\frac{3}{15}\bullet \circ H)||\frac{4}{15}\bullet \circ H)) = \frac{14}{30}\bullet \circ H$$

5.3 Copying

With $3 \circ_d P$ we denote three copies of P in the direction of dimension d. For a homogeneous object P of block shape with d as one of its directions it is plausible that

$$3 \circ_d \frac{1}{3}\bullet \circ_d P = \frac{3}{3}\bullet \circ_d P$$

and

$$2 \circ_d \frac{1}{4}\bullet \circ_d P = \frac{2}{4}\bullet \circ_d P$$

In combination with pseudo fracpairs and under the assumption that P is splitfree (not consisting of parts) we find:

$$2 \circ_d \frac{4}{\star}\bullet \circ_d 6 \circ_d P = 8 \circ_d P$$

and

$$\frac{4}{\star}\bullet \circ_{d,\mathsf{glue},-}\ 8 \circ_d \frac{1}{8}\bullet \circ_d P = \frac{1}{2}\bullet \circ_d P$$

6 More on lengths of linear blocks

We imagine a line l_{unit} of unit size, which constitutes a single object (i.e. it has not yet been split) is aligned in dimension d. Instead of $\equiv_{\mathsf{volume}}$ we write $\equiv_{\mathsf{length}}$ for this case. Now for instance

$$\frac{2}{2}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}} \neq l_{\mathsf{unit}}, \frac{2}{2}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}} \equiv_{\mathsf{length}} l_{\mathsf{unit}}$$

$$(\frac{2}{3}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}})||_{\mathsf{d}}(\frac{1}{3}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}}) \equiv_{\mathsf{length}} 3 \circ_{\mathsf{d}} (\frac{1}{3}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}}) \equiv_{\mathsf{length}} l_{\mathsf{unit}}$$

and

$$l_{\mathsf{unit}}||_{\mathsf{d}}l_{\mathsf{unit}} = 2 \circ_{\mathsf{d}} l_{\mathsf{unit}}$$

More generally, for natural a and b:

$$(a \circ_{\mathsf{d}} l_{\mathsf{unit}})||_{\mathsf{d}}(b \circ_{\mathsf{d}} l_{\mathsf{unit}}) = (a+b) \circ_{\mathsf{d}} l_{\mathsf{unit}}$$

So we are looking at $-||_{\mathsf{d}}-$ compositions of combined parts of l_{unit}. Here, if one were to take l_{unit} for a physical entity, different such units (supposedly of the same length) may be used. We recall that for a composition X, $\frac{a}{b}\bullet \circ_{\mathsf{d}} X$ applies $\frac{a}{b}\bullet \circ_{\mathsf{d}}$ to each of the parts of X and then combines the result in the same order. $\frac{a}{\star}\bullet \circ_{\mathsf{d}} X$, however takes the first a parts of the entire sequence, irrespective of the size of the different parts, so that for instance:

$$\frac{4}{\star}\bullet \circ_{\mathsf{d}} ((\frac{3}{3}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}})||_{\mathsf{d}}(\frac{2}{5}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}})) = (\frac{3}{3}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}})||_{\mathsf{d}}(\frac{1}{5}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}})$$

and

$$\frac{n+3}{\star}\bullet \circ_{\mathsf{d}} ((\frac{3}{7}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}})||_{\mathsf{d}}(X) = (\frac{3}{7}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}})||_{\mathsf{d}}(\frac{n}{\star}\bullet \circ_{\mathsf{d}} X)$$

Now we may define:

$$\mathsf{count}_{\mathsf{parts}}(\frac{2}{3}\bullet \circ l\mathsf{unit}) = 2$$

which generalizes to arbitrary fracpairs: $\mathsf{count}_{\mathsf{parts}}(\frac{n}{m}\bullet \circ_{\mathsf{d}} l\mathsf{unit}) = n$ and to arbitrary compositions:

$$\mathsf{count}_{\mathsf{parts}}(X||_{\mathsf{d}}Y) = \mathsf{count}_{\mathsf{parts}}(X) + \mathsf{count}_{\mathsf{parts}}(Y)$$

. One also finds for non-negative naturals n that

$$\mathsf{count}_{\mathsf{parts}}(\frac{n}{\star}\bullet) = \min(n, \mathsf{count}_{\mathsf{parts}}(X))$$

6.1 Length as a fracpair

We may assume that the length of l_{unit} equals 1 and drop the unit line as a feature of the formulae. This is a significant abstraction step because the reference to entities which are counted and measured gets lost. Fracpairs can be used to measure length. Though we will see that fracpairs when used as a measure of length will still preserve some information about how an object has been grouped into parts and in particular we define:

$$\mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(\frac{2}{3}\bullet \circ_{\mathsf{d}} l\mathsf{unit}) = \frac{2}{3}\bullet$$

which generalizes to arbitrary fracpairs: $\mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(\frac{n}{m}\bullet \circ l\mathsf{unit}) = \frac{n}{m}\bullet$. For objects H, K we find

$$H \equiv_{\mathsf{length}} K \iff \mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(H) \equiv \mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(K)$$

and

$$\mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(H) \equiv \mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(\frac{n}{n}\bullet \circ_{\mathsf{d}} H)$$

with $\equiv$ the following equivalence on fracpairs (including peripheral fracpairs):

(i) for integers a, b, $n \neq 0$ and $m \neq 0$, $\frac{a}{n}\bullet \equiv \frac{b}{m}\bullet \iff a \cdot m = b \cdot n$,
(ii) for integers a, b, $n \neq 0$, $\frac{a}{n}\bullet \not\equiv \frac{b}{0}\bullet$
(iii) for fracpairs X and Y, $X \equiv_{\mathsf{bs}} Y \rightarrow X \equiv Y$

The following rule is plausible:

$$\mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(H||_{\mathsf{d}}K) \equiv \mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(H) +_{\mathsf{B}} ||_{\mathsf{d}}\, \mathsf{length}^{\mathsf{fp}}_{\mathsf{B}}(K)$$

The above rule, however, presupposes that one knows how, i.e. in what order, compositions have been composed from smaller compositions composed. so that $(X||_{\mathsf{d}}Y)||_{\mathsf{d}}Z$ and and $X||_{\mathsf{d}}(Y||_{\mathsf{d}}Z)$ are distinguished.

$$(\frac{a}{n}\bullet \circ_{\mathsf{d}} H)||\frac{b}{n}\bullet \circ_{\mathsf{d}} H) = \frac{a+b}{n}\bullet \circ_{\mathsf{d}} H$$

We may define $\circ$ without dimension on fracpairs as follows (for integers a, b, n, and m):

$$\frac{-a}{\star}\bullet \circ X = -(\frac{a}{\star}\bullet \circ X) = \frac{a}{\star}\bullet \circ -X$$
$$\frac{a}{n}\bullet \circ \frac{b}{m}\bullet = \frac{a}{n}\bullet \cdot \frac{b}{m}\bullet$$

and, more abstracly and purely within fracpair calculus: assuming $a \geq 0$, $b \geq 0$, $m \geq 0$:

$$\frac{a}{\star}\bullet \circ \frac{b}{m}\bullet = \frac{\min(a,b)}{m}\bullet$$

6.2 C-addition

There is a third form of addition for fracpairs which is of use in the setting of measuring lengths of compositions. For positive naturals n and m we define $\mathsf{bridge}(n,m)$ as the smallest natural k such that $m \cdot k$ is a multiple of n, we further assume that $\mathsf{bridge}(n,0) = 0$, $\mathsf{bridge}(n,-m) = -\mathsf{bridge}(n,m) = \mathsf{bridge}(-n,m)$, and $\mathsf{bridge}(n,\bot) = \mathsf{bridge}(\bot,n) = \bot$. We take $\mathsf{bridge}(n,m)$ for a bridge from m to multiples of n. An important property of bridge is:

$$m \cdot \mathsf{bridge}(n,m) = n \cdot \mathsf{bridge}(m,n)$$

Now we define

$$\frac{\star}{n}\bullet \circ_{\mathsf{d}} (\frac{a}{m}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}}) = \frac{a \cdot \mathsf{bridge}(n,m)}{m \cdot \mathsf{bridge}(n,m)}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}}$$

and

$$\frac{\star}{n}\bullet \circ \frac{a}{m}\bullet = \frac{a \cdot \mathsf{bridge}(n,m)}{m \cdot \mathsf{bridge}(n,m)}\bullet$$

Now $+_{\mathsf{C}}$ can be defined as:

$$\frac{a}{n}\bullet +_{\mathsf{C}} \frac{b}{m}\bullet = (\frac{\star}{m}\bullet \circ \frac{a}{n}\bullet) +_{\mathsf{B}} (\frac{\star}{n}\bullet \circ \frac{b}{m}\bullet)$$

6.2.1 Measuring lengths in terms of fracpairs II

Fracpairs when used as a measure of length in a different manner will not depend on preserving information about how an object has been grouped if we define:

$$\mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(\frac{2}{3}\bullet \circ_{\mathsf{d}} l\mathsf{unit}) = \frac{2}{3}\bullet$$

which generalizes to arbitrary fracpairs: $\mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(\frac{n}{m}\bullet \circ l\mathsf{unit}) = \frac{n}{m}\bullet$. For composed objects, however the definition is adapted as follows:

$$\mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(H||_{\mathsf{d}}K) \equiv \mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(H) +_{\mathsf{C}} \mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(K)$$

Now one finds:

$$\mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}((X||_{\mathsf{d}}Y)||_{\mathsf{d}}Z) = \mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(X||_{\mathsf{d}}(Y||_{\mathsf{d}}Z))$$

For objects H, K we find

$$H \equiv_{\mathsf{length}} K \iff \mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(H) \equiv \mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(K)$$

and

$$\mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(H) \equiv \mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(\frac{n}{n}\bullet \circ_{\mathsf{d}} H)$$

6.2.2 Measuring length in terms of fracpairs III

One may contemplate a measure of length $\mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}$ which uses

$$\mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}(H||_{\mathsf{d}}K) \equiv \mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}(H) +_{\mathsf{C}} \mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}(K)$$

Now will not find:

$$\mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}((X||_{\mathsf{d}}Y)||_{\mathsf{d}}Z) = \mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}(X||_{\mathsf{d}}(Y||_{\mathsf{d}}Z))$$

In this case it is plausible to adopt:

$$\mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}(\frac{2}{3}\bullet \circ_{\mathsf{d}} l\mathsf{unit}) = \frac{3}{3}\bullet \circ \frac{2}{3}\bullet$$

in view of $\frac{2}{3}\bullet \circ_{\mathsf{d}} l\mathsf{unit} = \frac{1}{3}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}}||_{\mathsf{d}}\frac{1}{3}\bullet \circ_{\mathsf{d}} l_{\mathsf{unit}}$, and also:

$$\mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}(\frac{3}{5}\bullet \circ_{\mathsf{d}} l\mathsf{unit}) = \frac{5}{5}\bullet \circ \frac{5}{5}\bullet \circ \frac{2}{3}\bullet$$

In general we will find:

$$\mathsf{length}_{\mathsf{A}}^{\mathsf{fp}}(X) \equiv \mathsf{length}_{\mathsf{B}}^{\mathsf{fp}}(X) \equiv \mathsf{length}_{\mathsf{C}}^{\mathsf{fp}}(X)$$

6.3 D-addition

A further form of addition, $- +_{\mathsf{D}} -$ is found by working as follows: in order to determine $\frac{x}{y}\bullet +_{\mathsf{D}} \frac{u}{v}\bullet$ first calculate $\frac{x}{y}\bullet +_{\mathsf{C}} \frac{u}{v}\bullet = \frac{p}{q}\bullet$, then in case both numerator (p) and denominator (q) are integers (in this case DNN-integers) and the denominator is non-zero, simplify the resulting fracpair as follows: find integers r and s so that (i) $\frac{r}{s}\bullet \equiv \frac{p}{q}\bullet$ and (ii) $s > 0$ and (iii) s is minimal given requirements (i) and (ii). We will now have for instance $\frac{1}{4}\bullet +_{\mathsf{D}} \frac{1}{4}\bullet = \frac{1}{2}\bullet$ while $\frac{1}{4}\bullet +_{\mathsf{A}} \frac{1}{4}\bullet = \frac{8}{16}\bullet$, and $\frac{1}{4}\bullet +_{\mathsf{B}} \frac{1}{4}\bullet = \frac{1}{4}\bullet +_{\mathsf{C}} \frac{1}{4}\bullet = \frac{2}{4}\bullet$. D-addition enjoys the following properties.

Proposition 6.1. *For all simple fracpairs X and Y,*
$X \equiv Y \iff X +_{\mathsf{D}} \frac{0}{1}\bullet = Y +_{\mathsf{D}} \frac{0}{1}\bullet$

and

Proposition 6.2. *For all simple fracpairs X_1, X_2, Y_1 and Y_2,*
$X_1 +_{\mathsf{D}} X_2 \equiv Y_1 +_{\mathsf{D}} Y_2 \iff X_1 +_{\mathsf{D}} X_2 = Y_1 +_{\mathsf{D}} Y_2$

7 Concluding remarks

This work fits in a line of research where we attempt to discuss in detail issue which are often left implicit in accounts of EAwD. We mention Bergstra 2022 [7] where timing/phasing aspects of division are discussed, Bergstra 2024 [8] where the syntax of EAwD is discussed in detail, and Bergstra 2024 [9] where the role of explicit errors (as denoted by $\perp$) in EAwD is discussed.

We consider the use of new terminology in particular fracpair, justified because it avoids terminology like ratio and proportion all of which comes with expectations and subconcepts which may be confusing for readers and which may not be in line with our use of fracpair and fracpairing. For an informative account of different meanings of ratio we refer to Clark, Berenson & Cavey 2003 [22]. A (geometric) proportion is often understood as an equality between ratios, for instance $\frac{1}{3} = \frac{2}{6}$. Equations like $\frac{1}{3}\bullet = \frac{2}{6}\bullet$ are invalid in fracterm calculus whence an identification between fracpair and ratio seems to be unconvincing.

From a mathematical point of view there is nothing new in the proposed fracpair calculus. Euclidean geometry was developed almost 2500 year ago, making use of proportions rather than of rational numbers as constructed by R. Dedekind in 1887 [23] as infinite equivalence classes of pairs of integers. What we claim to be new is (i) the dedicated notation ($\frac{a}{b}\bullet$) for fracpairs, (ii) viewing fracpairs as unary operators (decorated versions of $\circ$) on compositions, (iii) various additional notations inclusive pseudo fracpairs ($\frac{a}{\star}\bullet$, $\frac{\star}{b}\bullet$), (iv) introducing operations on fracpairs irrespective of compatibility with various equivalence relations.

7.1 Minpairs

The classical definition of rational numbers on the basis of integers gives rise to contemplating fracpairs, and in a similar fashion one may deal with integers which are often understood as equivalence classes of pairs of naturals. Having available DNN-naturals we adopt a new sort Minpair with a constructor $- \multimap -$. Then for two naturals a and b, $a \multimap b$ is the minpair of a and b with as its most plausible interpretation $a - b$. The conventional equivalence on minpairs is as follows:

$$a \multimap b \equiv c \multimap d \iff a + d = c + b$$

Minpairs may also be given a mechanical interpretation in a world consisting merely of consecutive blocks, allowing a nearly trivial form of quantitative mereology, with $-||-$ as the juxtaposition operator for block sequences. Then, with X a sequence of blocks,

$$(5 \multimap 7) \circ X$$

stands for: (i) remove the 7 first blocks, and if there are too few blocks the operation crashes, say producing $\perp$, representing a state of crash, and (ii) if there was no crash, then add to the left 5 new blocks. Thus

$$(2 \multimap 3) \circ (B||B||B||B) = B||B||B$$

and $(1 \multimap 4)(B||B) = \perp$ and $(2 \multimap 2)(\perp) = \perp$, and one finds equations like

$$(6 \multimap 4) \circ (2 \multimap 5) \circ (B||B||B||B||B||B) = (8 \multimap 7) \circ (B||B||B||B||B||B) = \perp$$

Composition of $\circ$ may be defined on minpairs taken for operators on block sequences. We find double usage for $\circ$, as a composition operator and as an application of minpairs to block sequences. Said ambiguity of $\circ$ we take to be unproblematic. Composition then works as follows:

$$((a \multimap b) \circ (c \multimap d)) \circ X = (a \multimap b) \circ ((c \multimap d) \circ X)$$

Many equations result concerning equality (on all arguments, i.e. block juxtapositions) of compositions:

$$(a \multimap c) = (a \multimap b) \circ (b \multimap c)$$

$$((a + d) \multimap c) = (a \multimap b) \circ ((b + d) \multimap c)$$
$$a \multimap b = (a \multimap 0) \circ (0 \multimap b)$$
$$(a + b) \multimap c = (a \multimap 0) \circ (b \multimap c)$$
$$(a \multimap (b + c) = (a \multimap b) \circ (0 \multimap c)$$

The monus operation, $\dot{-}$, satisfies: $((n + 1) \dot{-} (m + 1) = n \dot{-} m, n \dot{-} 0 = n, 0 \dot{-} n = 0)$. Using monus we find:

$$(a \multimap b) \circ (c \multimap d) = (a + (c \dot{-} b)) \multimap (d + (b \dot{-} c))$$

Minpairs are useful for giving a precise account of arithmetic at a lower level of abstraction, as well as an earlier stage of development where integers have not yet been taken as the standard point of departure.

7.2 The fraction definition problem restated

The fraction definition problems was stated in Bergstra 2020 [5], and then the main conclusion was that no definition had been found. However, the issue seems to be in need of a more precise formulation and we restate the fraction definition problem as follows, in parts (A) and (B):
(A) determine which of the following is the case:

1. fraction is a concept which admits an unambiguous definition,
2. fraction fails to admit an unambiguous definition and is an ambiguous category (using the terminology of Brubaker 2025 [21]),
3. fraction is a blended notion which combines different mutually incompatible concepts into a single meaning, where one concept serves as the substance (essence, the real thing), and another concept serves as the outer appearance (a typical example of blending reads: "a fraction is a fracvalue written as a fracterm". Here the fraction is not merely a fracvalue, a fraction is in addition to a fracvalue also a fractoken).
4. fraction is both a productively ambiguous in the terminology of [21], and it is a category which admits blending for one or more of its alternative meanings,
5. The situation is more complex and it is part of the design (intended meaning) of fraction that (a) different people assign different meanings to it, (b) different persons may have different polices for ambiguation (using ambiguous language where unambiguous language is an option) and disambiguation, (c) for a single person the policies of (dis)ambiguation and resolution of blending may differ at different times
6. None of the above.

(B) in case 1 provide a definition for fraction, in case 2 provide a description for fraction, in both other cases: explain details.

From [5] we adopt arguments for the claim (found in Fandino Pinilla 2007 [26] and stated in different wording in Wu 2011 [29]) that Case 1 does not apply: fraction is not a mathematical concept which admits an unambiguous definition. Thus fraction may be an ambiguous category, or it may be a blended category (which we know little about), or ambiguity may be combined with blending. We are unable to make definite progress in the matter by having some theory of fracpairs at our disposal. However, we may consider which implications a focus on fracpairs may have for the fraction definition problem. We assume that DNN-naturals and DNN-integers are known concepts. DNN stands for decimal natural number and the idea is that, say the sign 23 is itself a natural number, and that DNN-integers are either DNN-naturals (including 0) or signs of the form $-p$ with p a positive DNN-natural (we refer to Bergstra 2020 [6] for a justification of these conventions.

Now we assume that a fracpair over DNN-integers is an unproblematic notion in the sense that from a point of view of semantics it is merely a pair of values which, upon formalization in many-sorted FOL deserves its own sort. We see that: (i) fracpair is a key landmark notion for fraction and (ii) there is no (significant) fracpair definition problem. while (iii) fracpair calculus comes with an incentive for using a more abstract notion of quantity (number).

Claim 7.1. *At each level of mathematical maturity an approach based on fracpairs with a spectrum of operators and equivalences is feasible. The fraction definition problem, however, arises because of dissatisfaction with fracpair calculus, and a perceived need to (i) work modulo quantitative equivalence ($\equiv$), and (ii) to qualify the resulting equivalence classes as numbers which are subsumed under the "universal" sort number so that* $2 = [\frac{2}{4}\bullet]_{\equiv} = \frac{2}{4}\bullet/\equiv$, *and (iii) to write* $\frac{2}{4}$ *instead of* $[\frac{2}{4}\bullet]_{\equiv}$.

From an operational point of view, a fracpair provides for a range of actions of splitting and selection, where the case that the operand is a number is an afterthought rather than a primary intuition. However, one may contemplate the following notations:

$$\frac{2}{3}\bullet \circ 1 = 2 \circ (\frac{1}{3}\bullet \circ 1)$$

and one may experiment with the idea that $\frac{2}{3}$ may have $\frac{2}{3}\bullet \circ 1$ as its meaning. Our idea is then that making sense of an expression

$$\frac{a}{b}\bullet \circ c$$

for DNN-integers a, b, and c may begin in the absence of a notion of rational number and may merely be based on a primary focus on, and understanding of fracpairs. We may view $\frac{2}{3}\bullet \circ 1$, $\frac{2}{1}\bullet \circ 3$, and $2 \circ (\frac{1}{3}\bullet \circ 1)$ as expressions, say collectively referred to as DNN fracpair application expressions, for which the meaning is yet to be understood. Initially such meaning assignment is given in mereological terms, while in a second stage, once an awareness of rational numbers has been acquired, the rational number $\frac{a \cdot c}{b}$ may be proposed as a proper semantics for $\frac{a}{b}\bullet \circ c$, where the latter meaning assignment provides additional motivation for the introduction of rationals.

7.3 Can fracpairing replace division?

One may ask to what extent the use of fracpairs may replace division in teaching elementary arithmetic, in other words can EAwFP be used instead of EAwD in an educational setting. At this stage we expect that such replacement is implausible. The overhead of having to keep track of different versions of addition, multiplication and inverse/division, as well as of different equivalence relations on frcpairs is just too cumbersome, and the transparency of working with a bundle of very clearly defined notions may not compensate for that overhead, and does not sufficiently improve on the synthetic fracterm calculus of Bergstra & Tucker 2024 [17].

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