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Review of Fibonacci's Liber Abaci by Sigler: The Loss of Division by Zero

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Abstract

We review "Fibonacci's Liber Abaci" by L. E. Sigler, 2002 CE. This is a literal English translation of a Latin text by Leonardo Pisano, better known as Fibonacci, dating back to 1202 CE. Fibonacci learned Hindu arithmetic and Hindu-Arabic algebra from Muslim North African Arabs and published a Latin treatise on it, incorporating his own mathematical developments. This text was highly influential in Europe. Fibonacci used a placeholder zero and had some knowledge of its algebraic structure, suggesting the re-emergence of an algebraic number zero. However, he seems to have had no direct contact with the works of Brahmegupta, 628 CE, who we credited with defining a total paraconsistent arithmetic of fractions that includes negative numbers and division by zero, nor of Bhascara, 1150 CE, who we credited with defining a total consistent version of Brahmegupta's fractional arithmetic. Fibonacci gave multiple definitions of division, but these are vague. He allows, but does not claim, a total paraconsistent division of non-negative fractions and a total inconsistent division of non-negative decimal integers. However, these inconsistencies would dissolve if Fibonacci did not regard zero as a number. The numberhood of zero is, therefore, critical to a proper understanding of Fibonacci's arithmetics.

Fibonacci gives a Euclidean proof of the correctness of the division of fractions that depends on their corresponding ratios, but he did not notice the corollary that division by zero exists because multiplication by zero exists. Thus, the Euclidean mathematics, of about 300 BCE, supported multiplication and division by zero as soon as zero became available in 628 CE! However, Fibonacci does not mention division by zero or negative numbers so, presumably, he did not learn Brahmegupta's or Bhascara's fractional arithmetic, whence division by zero was lost until modern times.

1 Introduction

In the book *Fibonacci's Liber Abaci*, 2002 CE, [11], Laurence Sigler (deceased) has an address at Mathematics, Bucknell University, Lewisburg, PA 17837. His book is a literal English translation and correction of a collated version of Latin examples of Leonardo Pisano's *Liber Abbaci* (spelled with a double b) published in Latin by Baldassarre Boncompagni in 1857 CE. The title Liber Abaci/Abbaci means *Book of Calculation*. Henceforth, we use the spelling *Abaci* with a capital *A* and a single *b*.

Leonardo Pisano is better known as Fibonacci. In 1202 CE, he published the first edition of his Liber Abaci in Latin. A revised edition appeared in 1228 CE. It is remarkable that it took eight hundred years for a translation to appear in a living language, for which translation Sigler deserves posthumous praise.

Sigler records that Fibonacci learned the Hindu numerals and decimal place-value system, along with a fractional arithmetic and some algebra, from Muslim North African Arabs. But on reading Sigler's translation we find no textual hint that Fibonacci had direct contact with Brahmegupta's 628 CE seminal Sanskrit works that established zero as an algebraic number in what we have credited with being a total but paraconsistent arithmetic of fractions that includes division by zero [4], [9]. Nor can we find any textual hint that Fibonacci was aware of Bhascara's 1150 CE improvement to Brahmegupta's arithmetic that we have credited with being a total consistent arithmetic of fractions that includes division by zero [4], [9]. It would therefore seem that division by zero was not transmitted to Fibonacci by Arabs at the end of the twelfth century, despite this knowledge being maintained in active Sanskrit scholarship until at least 1621 CE [4], [9]. Thereafter, under the influence of the British Raj, Sanskrit fell out of use as a scientific lingua franca, pausing transmission of division by zero to the English speaking world until Colebrooke's translation of Sanskrit arithmetic published in 1817 CE [9] and our 2026 CE analysis of it [4].

Nonetheless, we do find that some of Fibonacci's methods are identical to those of Brahmegupta and Bhascara. When we notice a precise coincidence of methods, we cite the earliest author – usually Brahmegupta.

According to Sigler, Fibonacci's text contains much of the mathematics known in Europe at that time, together with proofs using Euclidean geometry. The diligent reader can, therefore, master this Medieval mathematics, together with its Classical foundations, by reading a few texts, especially with the assistance of Sigler's introduction, where he explains Fibonacci's methods in today's mathematical notation. Sigler's appendix also gives some worked answers, in today's notation, to problems set by Fibonacci. Even the casual reader can appreciate much of Medieval commerce and daily life by a cursory reading of Fibonacci's problems.

As fascinating as Fibonacci's book is, we have a more narrow interest. We would like to know at what stage Arabia lost the knowledge of division by zero. That is a question we cannot answer here, but we do examine the mathematical question of how Fibonacci defined division so that division by zero became inaccessible.

1.1 Kindle Book

We have a Kindle Book version of Sigler's text [11]. This has the advantage of being able to choose the size of text and to turn pages and search quickly. It also has the very great advantage of being able to copy and paste English text accurately. However, there are some problems with the Kindle Book that make it unhelpful as a scholarly source.

Firstly, the copy mechanism does not work on some pages.

Secondly, the search mechanism ignores punctuation. In other words, it ignores mathematical symbols. This is unhelpful in a 638 page book of mathematics, even though much of the presentation is verbal, reflecting the structure of Fibonacci's original.

Thirdly, the search results are presented for the whole book, without allowing the reader to choose a range of pages in which to search. This is exceedingly tedious in a 638 page book.

Fourthly, the dynamically generated contents menu is vacuous. This can be finessed by bookmarking the static contents page printed in the book and then bookmarking the current page so that the *annotations* menu can be used to quickly consult the contents page and then return to the current page. Bookmarking also provides a quick way to access chapters and appendices.

Fifthly, Sigler provides very helpful numbered chapter notes in Chapter 16. The notes are numbered anew for each chapter. Unfortunately the note citations do not say on which page the note text occurs and the note text does not say on which page the note citation occurs. Instead the reader must search for these manually. This is exceedingly tedious. The publisher could provide bidirectional hyperlinks. In their absence, we list the necessary information here.

We give the start page of each chapter, together with the name of the chapter or an indication of its contents, followed by three-tuples describing the notes by $(number, citation, text)$ where: *number* is the note number, m, or a run of note numbers, $m - n$; *citation* is the page on which the note is cited; *text* is the page on which the text of the note starts. For example: $(1 - 6, 15, 617)$ indicates that notes 1 to 6, inclusive, are cited on page 15, and each note's text starts or appears in its entirety on page 617.

- Chapter -1: Sigler's Introduction, starts on page 1. No notes.
- Chapter 0: Fibonacci's Dedication and Prologue, starts on page 15. Notes: $(1 - 6, 15, 617)$, $(7 - 8, 16, 617)$ but Note 8 redirects to Note $(7, 554, 631)$.
- Chapter 1: Positive Decimal Integers, starts on page 17. Notes: $(1 - 2, 17, 618)$, $(3 - 4, 20, 618)$.
- Chapter 2: Multiplication of Non-Negative Decimal Integers, starts on page 23. No notes.
- Chapter 3: Addition of Positive Decimal Integers, starts on page 39. Notes: $(1, 40, 618)$, $(2, 41, 618)$, $(3, 43, 618)$.
- Chapter 4: Subtraction of a Lesser Decimal Integer from a Greater Decimal Integer, starts on page 45. No notes.

- Chapter 5: Various Forms of Division, starts on page 49. Notes: $(1-2,49,618)$, $(3-6,50,619)$ but the citation of Note 6 is missing from page 50, $(7,53,619)$, $(8-9,57,619)$, $(10,59,619)$, $(11-12,61,619)$, $(13,64,619)$, $(14,66,619)$, $(15,70,620)$, $(16,76,620)$.
- Chapter 6: Multiplication of Fractions, begins on page 77. Notes: $(1,77,620)$, $(2,77,620)$ Note 2 redirects to $(4,50,619)$, $(3,83,620)$.
- Chapter 7: Addition, Subtraction, and Division of Fractions, begins on page 99. Notes: $(1-3,119,620)$, $(4,121,620)$, $(5,123,620)$, $(6,126,620)$.
- Chapter 8: Proportions of Quantity and Price, begins on page 127. Notes: $(1-11,127,621)$, $(12-18,128,621)$, $(19,132,621)$, $(20,134,621)$, $(21,138,621)$, $(22,140,622)$, $(23-24,142,622)$, $(25,143,622)$, $(26,147,622)$, $(27,156,622)$, $(28-30,157,622)$, $(31-33,158,622)$, $(34,159,622)$, $(35,161,622)$, $(36,164,622)$, $(37,167,622)$ but the citation is missing, presumably from page 167, $(38-40,168,622)$, $(41,170,622)$, $(42-43,171,622)$, $(43-44,172,622)$ but the citations are misnumbered, they should be Notes $44-45$, $(46-48,175,622)$, $(49,176,622)$, $(50-51,177,622)$.
- Chapter 9: Proportions in Fair Barter by Price, begins on page 179. Notes: $(1-2,179,623)$, $(3,180,623)$, $(4,211,623)$.
- Chapter 10: Proportions in Fair Shares of Profit and Loss, begins on page 213. Notes: $(1,213,623)$.
- Chapter 11: Linear Equations and Inequalities in the Alloying of Precious Metals and Goods, begins on page 227. Notes: $(1,234,623)$, $(2,249,623)$, $(3,250,624)$.
- Chapter 12: Miscellany Including Sums of Series, Linear Equations, and Breeding Rabbits, begins on page 259. Notes: $(1,266,624)$, $(2-3,268,624)$, $(4,269,625)$, $(5,271,625)$, $(6,290,625)$, $(7,291,625)$, $(8,299,625)$, (9,301,626), (10-11,313,626), (12-13,314,626), (14,315,626), (15,322,626), (16,336,626), (17,337,626), (18-19,341,626), (20,347,626), (21,380,626), (22,399,626), $(23,401,627)$, $(24-26,404,627)$, $(27,419,627)$, $(28-29,428,627)$, $(30,429,628)$, $(31,439,628)$.
- Chapter 13: General Linear Equations (Elchataym), begins on page 447. Notes: $(1,447,628)$, $(2,449,629)$ the text of Note 2 is misformatted, it appears in the text of Note 1, $(3,449,629)$, $(4,450,630)$, $(5,456,630)$ but the citation is misnumbered as Note 4, $(6,463,630)$, $(7,468,630)$, $(8,472,630)$, $(9,484,630)$.
- Chapter 14: Square and Cube Roots, begins on page 489. Notes: $(1,489,630)$, $(2,494,630)$, $(3,510,631)$.
- Chapter 15: Quadratic Equations, begins on page 531. Notes: $(1,543,631)$, $(2,549,631)$, $(3-4,551,631)$, $(7,554,631)$, $(8-10,554,632)$, $(11,567,632)$, $(12,571,632)$, $(13,588,632)$, $(14,596,632)$, $(15,597,632)$, $(16,598,632)$, $(17,599,633)$, $(18,600,633)$, $(19,609,633)$, $(20,611,633)$.
- Chapter 16: Sigler's Notes for Liber Abaci, begins on page 617.
- Chapter 17: Sigler's Bibliography, begins on page 635.

1.2 Pragmatic Translation

In previous work [4] we described our theory of pragmatic translation. Sigler provides a translation of Liber Abaci into modern English so there is no difficulty reading his text. However, Sigler provides a literal translation and we want a pragmatic translation that expresses the mathematical content of Liber Abaci in modern mathematical symbols. We want to find the mathematical jigsaw pieces that were found, lost, and found again as the modern picture of mathematics emerged. We are much less concerned with the history of how mathematics was understood and used in different places, times, and languages.

We restrict ourselves to a consideration of the algebraic properties of the four fundamental operations of arithmetic: addition, subtraction, multiplication, division. We compare them to the earlier Sanskrit arithmetics of Brahmegupta and Bhascara [4], [9].

Our pragmatic translation depends on two principles. The *Principle of Generosity* states that we should give a coherent mathematical translation wherever possible. The *Principle of Least Commitment* says we should always give conservative mathematical translations. These two principles are deliberately set in tension with each other to encourage the most accurate translation, consistent with mathematical coherence.

Fibonacci used a few Arabic words where Latin had no equivalent. Sigler continued the use of these Arabic words. We make little use of these words in our paper, but we explain them to assist the independent reader.

The word *almuchabala* means *algebraic simplification.*

The word *elchataym* means the *method of double false position* for obtaining one or many solutions by general linear interpolation that passes through a possibly zero offset from the origin. This method solves the matrix equation $Ax + B = C$.

The word *hasam* means *prime number.*

The name *Maumeht* means *al-Khwarizmi*, the celebrated Arab algebraist.

Fibonacci uses the word *zephir* for the numeral 0. Zephir is the Latin transliteration of the Arabic word, which is the pragmatic translation of the Sanskrit *sunya*, which Brahmegupta used [9], among other Sanskrit words, to refer to both the numeral 0 and the algebraic number 0. The Latin *zephir* is an ancestor of our word *zero.* Sigler uses Fibonacci's word *zephir,* not the modern word *zero.*

Fibonacci writes about *numbers*, but his notion of what constitutes a number is vague and develops throughout his text. We prefix the modern vocabulary for numbers with the word *decimal* to identify Fibonacci's numbers and to give an indication of how they relate to modern numbers. For example, Fibonacci's *decimal integers* have only some of the properties of modern *non-negative integers.*

Fibonacci describes the addition, subtraction, multiplication, and division of numbers. He seems to regard numbers as non-negative, though he demonstrates an implicit understanding of negative numbers that he does not make explicit anywhere in the Liber Abaci. One of his examples of sequential trades begins with an implied initial loss, a negative number, but the successive trades sum to a profit, a positive number, so that negative numbers never explicitly enter the calculation. See Siegler's note (15,322,626), [11].

Fibonacci's account of numbers is so vague that we are not sure if he regards zero as a number. If he does, then his division of decimal integers is inconsistent and his division of fractions is paraconsistent. We handle this by saying that Fibonacci *allows, but does not claim,* various things. For example, Fibonacci allows, but does not claim, division by zero.

Henceforth, we set quotations from Sigler's book [11] in *italics* and add parenthetic comments in Latin braces {like this}.

1.3 Analytical Tools

We adopt all of the analytical tools in Section 1.3 of [4], but we make little use of the mnemonic set out there. Instead, we take n, d, z as both Fibonacci's decimal and modern non-negative integers and we take p as both one of Fibonacci's decimal and modern positive integers. Thus, the letters n, d, z, p provide a correspondence between Fibonnaci's numbers and modern numbers.

We also adopt Bergstra's syntactic *fracterms,* introduced in 2015 CE [5], [6], [7], [8]. Fracterms are a two-tuple, $\frac{n}{d}$, of an arbitrary numerator, n, and an arbitrary denominator, d. The user adds algebraic structure, typically to define n, d, and $\frac{n}{d}$ as numbers.

Fibonacci describes arithmetical algorithms by speaking of the literal numbers in numerical examples. This makes it difficult to keep track of the algebraic roles of the specific numbers he chooses. We annotate this role by setting each number equal to a variable, as in the following example of $\frac{2}{3} \times \frac{5}{7} = \frac{2\times 5}{3\times 7} = \frac{10}{21}$.

$$\frac{(n_1 = 2)}{(d_1 = 3)} \times \frac{(n_2 = 5)}{(d_2 = 7)} = \frac{(n_1 = 2) \times (n_2 = 5)}{(d_1 = 3) \times (d_2 = 7)} = \frac{(n_3 = 10)}{(d_3 = 21)}$$

Now we can read off the algebraic roles as:

$$\frac{n_1}{d_1} \times \frac{n_2}{d_2} = \frac{n_1 \times n_2}{d_1 \times d_2} = \frac{n_3}{d_3}$$

We may add a subscript j to a variable v_i to give v_{ij}, where j denotes the j'th occurrence of the i'th variable. For example, in the sum $v_{11} + v_2 + v_{12}$, the first variable, v_{11}, is the first occurrence of v_1, the second variable, v_2, is the only occurrence of v_2, and the third variable, v_{12}, is the second occurrence of v_1. Thus, the j'th index gives the position of a variable in a formula. This indexing enables a concise discussion of variables in formulas.

2 Fibonacci's Arithmetic

In the Prologue of the second edition of the Liber Abaci, 1228 CE, Fibonacci writes of the first edition, 1202 CE: *In it I presented a full instruction on numbers close to the method of the Indians, [3] whose outstanding method I chose for this science.* Sigler's Note $(3, 15, 617)$ says that Fibonacci used the phrase *Indian figures* for the numerals zero to nine. Why then and when did these figures become known as "Arabic" numbers instead of Fibonacci's original "Indian" numbers? Be that as it may, we follow the modern practice of calling these the "Hindu-Arabic" numerals, reflecting the historical migration of the concept and calligraphy of the decimal numerals.

Fibonacci split the Liber Abaci into fifteen chapters, numbered in Roman numerals from I to XV inclusive. He gave no chapter titles, but listed the contents of each chapter at the beginning of that chapter. Here we follow Fibonacci in listing chapters by number, but we use Hindu-Arabic numbers instead of Roman numbers.

2.1 Chapter 1

In Chapter 1, starting on page 17 of Sigler's translation of Liber Abaci, Fibonacci writes:

The nine Indian figures are:

9 8 7 6 5 4 3 2 1.

With these nine figures, and with the sign 0 which the Arabs call zephir [1] any number whatsoever is written, as is demonstrated below.

Fibonacci acknowledges the *nine figures* as they are called in Sanskrit mathematics [9], but he regards 0 as a different kind of object – not a *figure* but a *sign*. This is a mistaking of mathematical history, written in Sanskrit, because in 628 CE [4], [9] zero was already an algebraic number taking part in addition, subtraction, multiplication, and division and the symbol $\circ$, similar to the modern day 0, was already in use to represent both the numeral zero and the algebraic number zero before 1150 CE [9]. Thus, in Sanskrit mathematics, the written predecessors of all of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 were considered to be both written numerals and algebraic numbers. Fibonacci's mistake speaks to the incomplete transmission of Sanskrit mathematics via Arabia to Fibonacci.

Fibonacci continues: *A number is a sum of units, or a collection of units, and through the addition of them the numbers increase by steps without end [2].*

Thus, Fibonacci says that the numbers are 1, 2, 3, ..., without end. Whence, by exclusion, 0 is not a number. Two ways of making zero a number come to mind. Firstly, we could give an axiom to declare that zero is a number. Secondly, we could give equalities that relate zero to a number, such as: $0 + 1 = 1$, $0 \times 1 = 0$, $1 - 1 = 0$. Then, by the closure of arithmetic, 0 would be a number.

Fibonacci continues: *First, one composes from units those numbers which are from one to ten. Second, from the tens are made those numbers which are from ten up to one hundred. Third, from the hundreds are made those numbers which are from one hundred up to one thousand. Fourth, from the thousands are made those numbers from one thousand up to ten thousand, and thus by an unending sequence of steps, any number whatsoever is constructed by the joining of the preceding numbers.*

This is almost the same as the modern place-value system, but there are differences in the handling of the algebraic number zero, that takes part in arithmetic, and leading zero numerals,that are used only to write a number.

In the modern place-value system, it is taught that the algebraic numbers 0 to 9 occur in the first place; a leading placeholder zero, that is not written, and the numbers 10 to 90 occur in the second place; a leading placeholder zero, that is not written, and the numbers 100 to 900 occur in the third place; and so on. But Fibonacci treats the numeral 0 as a non-leading placeholder and says that the first ten numbers are 1 to 10, with 10 being the carry into the next place. That is, Fibonacci lacks the algebraic number 0 in his place-value system, the numeral 0 entering only as a non-leading placeholder. In fact, he lacks all leading zero numerals, such as $01 = 1$ and $00 = 0$.

Fibonacci gives look-up tables for addition and multiplication. The lowest entry in these tables is two, 2. There is no indication, anywhere in this chapter, of how to add or multiply by zero, 0, or unity, 1, so the tables treat both 0 and 1 as placeholders. However, Fibonacci has said that 1 is a number, which leaves 0 as the only placeholder. Consequently we adopt the following definition, which we update in Chapter 2.

Our definitions make explicit certain mathematical facts that are only implicit in Fibonacci's text. That is to say, our definitions identify pragmatic mathematical meanings that are only implicit in Fibonacci's literal text. We update our definitions as new facts come to light later in Fibonacci's text.

Definition 2.1. *The positive decimal integers, $p \geq 1$ exist.*

2.2 Chapter 2

In Chapter 2, starting on page 23, Fibonacci describes multiplication of positive decimal integers and gives a method for checking such multiplications by casting out nines. Both the multiplication algorithm and the checking algorithm generally require some addition.

On page 27, Fibonacci correctly calculates $7 \times 308 = 2,156$. Midway through the calculation we read: *the* 7 *is multiplied by the* 0 *which makes* 0. That is, $7 \times 0 = 0$. Now, by the closure of arithmetic, zero is a number. We therefore expand our definition of decimal integers to include zero.

Definition 2.2. *The non-negative decimal integers, $z \geq 0$ exist.*

On page 26, Fibonacci calculates $70 \times 70 = 4,900$. But the zeros are not multiplied or added. Instead Fibonacci calculates $7 \times 7 = 49$ and then concatenates two zeros to the answer, as $49 \oplus 00 = 4,900$. Fibonacci uses erasure and concatenation of zeros in many products of zero, thereby avoiding the calculation $0 \times 0 = 0$. However, we generously assume that this shorthand use of placeholder zeros is motivated by Fibonacci's understanding of the multiplication of algebraic zeros.

Definition 2.3. *For all non-negative decimal integers, z, it is the case that $0 \times z = 0$.*

On page 25 we read, *the multiplication therefore of* 1 *by* 1*, namely* 1. Which is to say, $1 \times 1 = 1$.

On page 26 we read, *4 which is multiplied with the kept 1; there will be 4*. Which is to say $4 \times 1 = 4$. After many such examples, we conclude that Fibonacci understands multiplication of positive decimal integers by unity.

Definition 2.4. *For all positive decimal integers, p, it is the case that $1 \times p = p$.*

Now, Definition 2.3 and 2.4 together support the following definition.

Definition 2.5. *For all non-negative decimal integers, z, it is the case that $1 \times z = z$.*

We suppose that the general rules in Definition 2.3 and 2.4 were so well known, or so obvious from the multitude of examples in Liber Abaci, that there was no need to write entries for zero and unity in the tables for multiplication given in Chapter 1. Alternatively, if Fibonacci did not regard zero as a number, then there was no need to give tables for multiplication by algebraic zero, despite the fact that Fibonacci can correctly multiply by a placeholder zero.

We repeatedly return to the issue that it is not clear to us whether Fibonacci treated zero as an algebraic number or as a placeholder, and it may be that he treated zero in different ways, in different contexts.

In summary, we generously conclude that Fibonacci could multiply all combinations of non-negative decimal integers and that all of these non-negative decimal integers are algebraic numbers, albeit that they have less algebraic structure than modern integers. For example, Fibonacci orders the positive decimal integers and understands which fractions lie between given positive decimal integers, but he does not give an ordering of zero. Furthermore, Fibonacci has no explicit account of the arithmetic or ordering of negative decimal integers.

2.3 Chapter 3

In Chapter 3, starting on page 39, Fibonacci gives one algorithm for adding positive decimal integers and another for multiplying positive decimal integers.

Examining the tables for addition in Chapter 1, we see $10 + 10 = 20$. In the units place, this is $0 + 0 = 0$. We also see $2 + 10 = 12$. In the units place, this is $0 + 2 = 2$. Examining sums in the units place of the whole table, we learn how to add zero to every numeral, except unity. That is, we do not learn $0 + 1 = 1$ by examining the sums in the units place. Nonetheless, we generously assume that Fibonacci can add zero to every non-negative decimal integer.

Definition 2.6. *For all non-negative decimal integers, z, it is the case that* $0 + z = z$.

In the table for addition we see $10 + 10 = 20$. In the tens place, this is $1 + 1 = 2$, but we do not find any other sums with unity in this table. Nonetheless, we generously assume that Fibonacci can add unity to every non-negative decimal integer, for example, by counting on!

Definition 2.7. *For all non-negative decimal integers, z, it is the case that $z + 1$ exists.*

2.4 Chapter 4

In Chapter 4, starting on page 45, Fibonacci gives an algorithm for subtracting a smaller positive decimal integer from a larger positive decimal integer, whence the result is a positive decimal integer. Fibonacci describes how to perform multiplace subtraction and assumes that the reader can already perform subtraction of a single digit number from a larger number up to 18. Presumably, such simple subtractions were calculated by counting on. This might be settled by wider historical research.

Definition 2.8. *Let y and z be positive decimal integers, such that $z > y$. Then $z - y > 0$.*

Throughout Liber Abaci, Fibonacci gives many examples of negative numbers, which he calls *debits*, but he performs limited arithmetic on them. The eventual solution to his problems is always a sum of the form $z - y > 0$.

2.5 Chapter 5

In Chapter 5, starting on page 49, Fibonacci writes: *When one wishes to know how to divide any number by any number, it is necessary, as in adding, first to divide all numbers by the numbers from two up to ten; and this is not possible to do until certain introductions to divisions of certain* *numbers are known by heart; these divisions are given in tables in the following pages. But it first is taught how the small fractions are written.*

Fibonacci's concern *to know how to divide any number by any number* suggests he wants division to be total, but his tables give partial divisions of positive decimal integers where the divisor is at least two, 2. The tables allow an explicit quotient of zero, but do not allow an explicit remainder of zero. Instead the tables show a blank where a remainder of zero occurs. In other words, a zero quotient is given by an algebraic zero, whereas a zero remainder is given by a place-holder zero.

Fibonacci's divisions fall into two distinct classes: divisions with remainder and divisions without remainder. Neither of these integer divisions is a fraction, but Fibonacci eventually arrives at vulgar fractions, with no remainder, and mixed fractions, with a remainder. He calls the fractions that can be calculated from his tables *small* fractions, before generalising them to what we shall call *large* fractions. But Fibonacci's smallest denominator is two, so fractions with a denominator of unity or zero are not yet considered by him.

Fibonacci's tables support the following Definition 2.9, but the relationship between division and multiplication is not made explicit by Fibonacci until later. See Definition 2.15 below.

Fibonacci never explicitly states that the remainder is less then the divisor, $r < d$, but this is clear from his tables and examples. It seems to be an unstated convention.

Definition 2.9. *The division of a non-negative integer, n, by a positive integer, d, is given by non-negative integers $q,\ r$, such that $n \div d = (q, r)$, where $n = q \times d + r$ and $r < d$. Here q is called the quotient and r is called the remainder.*

When there is no remainder, that is when $r = 0$, Fibonacci writes the vulgar fraction: $n \div d = \frac{n}{d}$.

When there is a remainder, $r > 0$, Fibonacci writes either the vulgar fraction $n \div d = \frac{n}{d}$ or else he writes the equivalent mixed fraction $n \div d = \frac{r}{d}q$. Modern practice is to write mixed fractions with the fraction to the right of the integer, as $q\frac{r}{d}$.

Fibonacci then makes the totality of division explicit: *If over any number will be made a fraction line, and over the same line will be written another number, the upper number means the number of parts determined by the lower number; the lower is called the denominator and the upper is called the numerator.*

We are not sure whether Fibonacci intends to include zero as a *number.* However, Fibonacci gives multiple examples of $\frac{0}{d} = 0$ so zero must be an allowable numerator, hence, by what he has just said, zero is also an allowable denominator. Thus, we take Fibonacci as treating zero as a number. This runs into inconsistency, but this inconsistency is a consequence of Fibonacci's vagueness, which we can better handle once we understand the consequences, for Fibonacci's arithmetic, of allowing division by zero.

Definition 2.10. *The total division of all non-negative decimal integers n and d is given by $n \div d = \frac{n}{d}$.*

Remark 2.11. *At this stage, Fibonacci cannot identify the quotient of the fractions $\frac{p}{1}$, $\frac{0}{p}$, $\frac{0}{0}$, $\frac{p}{0}$ for any positive decimal integer p.*

Fibonacci continues: *And if over the number two will be made a fraction line, and over the fraction line the number one is written, then one of two parts of the whole is meant, that is, one half, thus $\frac{1}{2}$ ...* Fibonacci gives many such examples.

Now $\frac{n}{1}$ exists by Definition 2.10. Fibonacci does not define division by unity but, by what he has just been said, $\frac{n}{1}$ means n *parts of the whole* or n *wholes* or n.

Definition 2.12. *For all non-zero decimal integers, n, it is the case that $\frac{n}{1} = n$.*

Remark 2.13. *At this stage, Fibonacci cannot identify the quotient of the fractions $\frac{0}{q}$, $\frac{0}{0}$, $\frac{p}{0}$ for any positive decimal integers p and q such that $q > 1$.*

After giving several examples of vulgar fractions, Fibonacci defines composed (large) fractions of non-negative numerators, n_i, and positive denominators, d_i. In the definition, below, the numerators and denominators of a composed fraction are shown on the left hand side of the definition, $=_{\text{def}}$, where they are separated by horizontal spaces. Each pair of corresponding n_i and d_i is aligned vertically, with a single long horizontal rule separating pairs of n_i and d_i. The right hand side shows the implied sum of vulgar fractions. As the reader can see, this is a compact notation.

$$
\begin{aligned}
\frac{n_k \ n_{k-1} \ n_{k-2} \ \ldots \ n_1}{d_k \ d_{k-1} \ d_{k-2} \ \ldots \ d_1} =_{\text{def}} & \frac{n_k}{d_k \times d_{k-1} \times d_{k-2} \cdots \times d_1} + \\
& \frac{n_{k-1}}{d_{k-1} \times d_{k-2} \cdots \times d_1} + \\
& \frac{n_{k-2}}{d_{k-2} \cdots \times d_1} + \\
& \cdots + \\
& \frac{n_1}{d_1}
\end{aligned}
$$

Any or all of the n_i may be zero, such that $\frac{0}{p} = 0$ for all positive decimal integers p. We now give the following definition, which we later extend to Definition 2.18.

Definition 2.14. *For all positive decimal integers, p, it is the case that $\frac{0}{p} = 0$.*

As noted in Sigler's Introduction, on page 7, Fibonacci gives a decimal fraction, as a composed fraction with all denominators being 10, in Chapter 12, page 440.

In a plausible taxation problem, taken to an implausible extreme, Fibonacci computes the tax paid by a merchant visiting twelve cities as 28.2429536481 Bezants. Fibonacci gives this as the composed fraction:

$$\frac{1\ \ 8\ \ 4\ \ 6\ \ 3\ \ 5\ \ 9\ \ 2\ \ 4\ \ 2}{10\ 10\ 10\ 10\ 10\ 10\ 10\ 10\ 10\ 10}28$$

As Sigler notes, Fibonacci could use decimal fractions, but there was seldom any need, because decimal units were hardly ever used in the thirteenth century. After decimal calculation was popularised by Fibonacci and others, the use of decimal units increased.

In Chapter 5, Fibonacci goes on to describe different kinds of composed and mixed fractions.

On page 53 Fibonacci says of vulgar fractions: *And again it is noted that when any number* $\{n\}$ *is divided by another number* $\{d\}$*, then the multiplication of the quotient* $\{q = n \div d\}$ *and the divisor* $\{d\}$ *yields the number which is the dividend* $\{n = q \times d\}$*. Thus if 40* $\{n = 40\}$ *is divided by 4* $\{d = 4\}$*, then there results 10* $\{q = 10\}$*. Therefore if we multiply the 4* $\{d\}$ *by the 10* $\{q\}$*, then it makes forty* $\{n\}$*, namely the divided number* $\{d \times q = 4 \times 10 = 40 = n\}$. We generously assume that Fibonacci intends the following implication – but note, very carefully, that this is not a bi-implication!

Definition 2.15. *Let* n*,* d*,* q *be non-negative decimal integers, then* $n \div d = q \implies n = d \times q$*.*

The above implication says that we can obtain $n = d \times q$ from $n \div d = q$, which supports Definition 2.9, but it does not say we can obtain the converse $n \div d = q$ from $n = d \times q$.

Definition 2.15 is inconsistent in itself, as we show from Remark 2.19. But in our effort to explore the consequences, for Fibonacci's arithmetic, of allowing division by zero, we will later withdraw the inadequate definition 2.15 and replace it with an adequate Definition 2.21.

Remark 2.16. *At this stage, Fibonacci cannot identify the quotient of the fractions* $\frac{0}{0}$*,* $\frac{p}{0}$ *for any positive decimal integer* p*.*

We use Fibonacci's arithmetic to prove a special case of the Suppes-Ono convention, 1957 CE, [3], [12], which says that $\frac{r}{0} = 0$ for all real r. This convention is consistent with the modern algebraic field axioms.

Theorem 2.17. $\frac{0}{0} = 0$ *satisfies Fibonacci's specification of division in Definition 2.15.*

Proof. Suppose $\frac{0}{0} = 0$. By Definition 2.10 this is equivalent to $0 \div 0 = 0$. Then:

$$\begin{aligned} 0 &= 0 \times 0 \text{ by Definition 2.15} \\ &= 0 \text{ by Definition 2.3} \end{aligned}$$

Now $0 = 0$ is a true statement, which satisfies Definition 2.15. $\square$

Hence, we extend Definition 2.14

Definition 2.18. *For all non-negative decimal integers, z, it is the case that $\frac{0}{z} = 0$.*

Thus, Fibonacci allows, but does not claim, $\frac{0}{0} = 0$.

Remark 2.19. *At this stage, Fibonacci cannot identify the quotient of the fraction $\frac{p}{0}$, for any positive decimal integer p.*

There is an obstacle to identifying the above quotient. Suppose $\frac{p}{0} = q$, where p is a positive decimal integer and q is a non-negative decimal integer. Then $p = 0 \times q$ by Definition 2.15, whence $p = 0$ by Definition 2.3. But $p = 0$ contradicts the assumption that p is positive, whence Fibonacci's specifications of division are inconsistent if the quotient of $\frac{p}{0}$ is a non-negative decimal integer, q.

There are various ways forward. We could simply note the historical fact that Fibonacci gave inconsistent definitions of division, or we could generously make the definitions consistent. One way to remove the inconsistency would be to adopt the Suppes-Ono convention of 1957 CE [3], [12], but that would be egregiously anachronistic.

Recall that Fibonacci used the Hindu place-value system, so each of Fibonacci's decimal integers is similar to the corresponding decimal integer in Brahmegupta's and Bhascara's arithmetics. The Hindu authors had more algebraic structure in their number systems, including explicit negative numbers and an algebraic zero. Even though Fibonacci does not have this algebraic structure, we can use the Hindu *method* for removing the inconsistency of division by zero from Fibonacci's arithmetic. This is a more historically plausible way to remove the inconsistency.

Brahmegupta's division, specified in the Cuttaca, Verse 35 - 36, [4], [9], allowed division by zero by taking $\frac{0}{0} = 0$ and allowing each $\frac{n_i}{0}$ to be an irreducible fraction for all non-zero decimal integers and for all known computable irrational numbers, n_i. Bhascara's division, specified in the Vija-Ganita, Verse 14, [4], [9], took $\frac{0}{0}$ in the class of irreducible fractions, $\frac{n_i}{0}$, for all decimal integers and for all known computable irrational numbers, n_i. This makes Bhascara's division by zero consistent. Similarly, we can make Fibonacci's division consistent by disallowing $\frac{0}{0} = 0$.

Remark 2.20. *In order to make Fibonacci's division consistent, we withdraw our deduction that Fibonacci's arithmetic implies $\frac{0}{0} = 0$.*

We observe that Fibonacci prefigured division via the multiplicative inverse in Definition 2.15. However, as we have just seen, this definition is inconsistent. We generously replace Fibonacci's total inconsistent multiplicative inverse with the modern partial consistent multiplicative inverse. This is the actual historical rounte by which this inconsistency was handled, but we go further, we totalise Fibonacci's division at Remark 2.22.

Definition 2.21. *Let n, q be non-negative decimal integers, when p is a positive decimal integer, then $n \div p = q \iff n = p \times q$.*

Notice that we now have the guarding clause $p \neq 0$ and the implication is strengthened to a bi-implication. This modified version of Fibonacci's specification of division is just the usual definition of division employing the multiplicative inverse when the divisor is non-zero:

$$\begin{aligned} n \div p &= n \times p^{-1} \\ &= q \end{aligned}$$

With the bi-implication $q \times p = n$ being derived as:

$$\begin{aligned} q &= n \times p^{-1} \\ q \times p &= n \times p^{-1} \times p \\ &= n \end{aligned}$$

Remark 2.22. *At this stage our modification of Fibonacci's division is total. It is also consistent because the totality is partitioned into mutually distinct parts.*

All fractions $\frac{n}{d}$ exist for all non-negative n and d by Definition 2.10.

When $d \neq 0$ we have $n \div d = q \iff n = d \times q$ by Definition 2.21.

When $d = 0$ it is the case that $\frac{n}{d}$ is irreducible.

2.6 Chapter 6

In Chapter 6, starting on page 77, Fibonacci describes the multiplication of compound, mixed, and vulgar fractions. In every case, the calculation proceeds by obtaining vulgar fractions, multiplying them, and then presenting the product as the desired kind of fraction. On page 78, Fibonacci correctly calculates $\frac{1}{2}11 \times \frac{1}{3}22 = \frac{5}{6}256$. He describes the calculation in a tableau arranged by rows and columns in the Sanskrit style. Lines *beneath* and *above* are scratch spaces for intermediate calculations. The scratch spaces and the accompanying illustration of the tableau do not add to our understanding so we do not present them here. The interested reader may consult Sigler [11].

Recall that Fibonacci writes mixed and compound fractions to the left of the decimal integer. He says that the fraction lies *after* the integer so *after* means *to the left of* in his verbal account of calculations.

In the following example, each of the literal numbers is unique, so the reader can readily identify the numbers in Fibonacci's verbal description with the corresponding numbers in our algebraic transcription.

If one will wish to multiply 11 *and one half by* 22 *and one third, then he writes the greater number beneath the lesser, namely* $\frac{1}{3}22$ *beneath* $\frac{1}{2}11$ *as is shown here; next he makes halves of the* $\frac{1}{2}11$ *because the fraction part with the* 11 *is halves, which one makes thus:*

Having laid out the tableau, Fibonacci then reduces the mixed fraction $\frac{1}{2}11$ to the vulgar fraction $\frac{23}{2}$. This conversion is the same as Chaturveda's comment, before about 890 CE, on Brahmegupta's Ganita, Verse 2 [4], [9].

$$\begin{aligned}\frac{r}{d}q &= \frac{qd+r}{d}\\ \frac{(r=1)}{(d=2)}(q=11) &= \frac{(q=11)(d=2)+(r=1)}{(d=2)}\\ &= \frac{(n_1=23)}{(d_1=2)}\end{aligned}$$

you will multiply the 11 *by the* 2 *that is under the fraction line after the* 11*, and to this product you add the* 1 *which is over the fraction line over the* 2*; there will be* 23 *halves, or the double of* $\frac{1}{2}11$ *halves; there will be* 23*; you write the* 23 *above the* $\frac{1}{2}11$*, as is shown in the illustration;*

Next, Fibonacci reduces the mixed fraction $\frac{1}{3}22$ to the vulgar fraction $\frac{67}{3}$. We assume the reader can read off the algebraic structure from the numerical equalities. This structure is, in any case, just the structure presented above.

$$\begin{aligned}\frac{(r=1)}{(d=3)}(q=22) &= \frac{(q=22)(d=3)+(r=1)}{(d=3)}\\ &= \frac{(n_2=67)}{(d_2=3)}\end{aligned}$$

and for the same reason you multiply the 22 *by the fraction part under the fraction line, that is the* 3 *that is under the fraction line after the* 22*; there will be 66 thirds to which you add the* 1 *which is over the* 3*; there will be* 67 *thirds that you keep above the* $\frac{1}{3}22$*, and the* 67 *is the triple of* $\frac{1}{3}22$*;*

Next, Fibonacci multiplies the two vulgar fractions that he has just obtained. His calculation involves holding intermediate products in scratch spaces, so the calculation is not as transparent as our algebraic expression of it.

Again we write out the algebraic structure, before presenting the algebraic structure encoded with the numerical equalities. Hereafter, we use only the encoded form.

$$\frac{n_1}{d_1} \times \frac{n_2}{d_2} = \frac{n_1 \times n_2}{d_1 \times d_3} = \frac{n_3}{d_3}$$

$$\begin{aligned}\frac{(n_1 = 23)}{(d_1 = 2)} \times \frac{(n_2 = 67)}{(d_2 = 3)} &= \frac{(n_1 = 23) \times (n_2 = 67)}{(d_1 = 2) \times (d_2 = 3)} \\ &= \frac{(n_3 = 1541)}{(d_3 = 6)}\end{aligned}$$

and you will multiply the 23 *halves by the* 67 *thirds; there will be* 1541 *sixths which you divide by the fraction parts which are under the fraction lines of both numbers, namely the* 2 *and the* 3*; the division is made thus: you multiply the* 2 *by the* 3*; there will be* 6 *by which you divide the* 1541*;*

Next, Fibonacci converts the product from a vulgar fraction to a mixed fraction.

... the entire quotient for the sought multiplication will be $\frac{5}{6}256$ *as is demonstrated in the written illustration.*

After many such examples, we adopt the following definition. As usual, we read Fibonacci as allowing, but not claiming, that multiplication applies to fractions with zero denominators.

Definition 2.23. *For arbitrary non-zero decimal integers* n_1*,* n_2*,* d_1*,* d_2 *the multiplication,* $\times$*, of fracterms* $\frac{n_1}{d_1}$ *and* $\frac{n_2}{d_2}$ *is given by:*

$$\frac{n_1}{d_1} \times \frac{n_2}{d_2} = \frac{n_1 \times n_2}{d_1 \times d_2}$$

Our algebraic re-write of Fibonacci is the same as Brahmegupta's 628 CE fractional multiplication set out in the Ganita Verse 3, [4], [9], except that Brahmegupta can also explicitly multiply negative numbers.

Notice that Brahmegupta, Bhascara, and Fibonacci all reduce fractions to vulgar fractions before multiplying them. They may then convert the vulgar product to some preferred kind of fraction.

2.7 Chapter 7

In Chapter 7, starting on page 99, Fibonacci describes the addition, subtraction, and division of unit fractions, $\frac{1}{d}$. He then generalises these arithmetical operations to all vulgar fractions, $\frac{n}{d}$. A remarkable consequence of his proof of the correctness of the division of fractions, which proof employs Euclid's proportions, is the conditional statement that if multiplication by zero exists as multiplication by the fraction, $\frac{0}{x}$, then division by zero exists as multiplication by the reciprocal fraction $\frac{x}{0}$. See Corollary 2.26 below and the citation of the Ancient Greek algorithm of Anthyphairesis. Thus, the Euclidean mathematics, of about 300 BCE, supported multiplication and division by zero as soon as zero became available as a fraction in 628 CE!

On page 100, Fibonacci begins with a special algorithm for adding unit fractions $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, before presenting a general algorithm for the addition of vulgar fractions, using the commutative example $\frac{1}{4} + \frac{1}{3} = \frac{7}{12}$, which gives the same result, $\frac{7}{12}$. That is, Fibonacci implicitly demonstrates both the commutativity of addition and the correctness of the general algorithm relative to the special algorithm.

We transcribe Fibonacci's algorithms into algebra, using j indices to track the occurrences of each variable, and we write parenthetic comments in Latin braces {like this}.

Note that Fibonacci follows the Sanskrit style of writing fractions in a tableau, without using an addition sign. The word *above* refers to a scratch space written above terms in the tableau.

We begin with Fibonacci's special algorithm:

$$\frac{1}{(d_{11} = 3)} + \frac{1}{(d_{21} = 4)} = \frac{(d_{12} = 3) + (d_{22} = 4)}{(d_{13} = 3) \times (d_{23} = 4)}$$

If you will wish to add $\frac{1}{3}$ *and* $\frac{1}{4}$ *then we teach you to do this in two ways, first indeed according to the common way. You find a number for which* $\frac{1}{3}$ *and* $\frac{1}{4}$ *of it are integers; the number is found thus: you multiply the* 3 $\{d_{11} = 3\}$ *by the 4* $\{d_{21} = 4\}$ *that are under the fractions; there will be 12* $\{(d_{13} = 3) \times (d_{23} = 4) = 12\}$; $\frac{1}{4}$ *and* $\frac{1}{3}$ *of it are found; and then you take a third of it that is* 4 {*calculated most directly as* $d_{11} = 3$}, *and a fourth of it that is* 3 {*calculated most directly as* $d_{21} = 4$}, *and you add them together; there will be 7* $\{(d_{12} = 3) + (d_{22} = 4) = 7\}$ *that you divide by the 12; the quotient will be* $\frac{7}{12}$, *that is seven of twelve parts of the unity.*

Now we present Fibonacci's general algorithm:

$$\frac{(n_{11} = 1)}{(d_{11} = 4)} + \frac{(n_{21} = 1)}{(d_{21} = 3)} = \frac{(n_{22} = 1) \times (d_{12} = 4) + (n_{12} = 1) \times (d_{22} = 3)}{(d_{13} = 4) \times (d_{23} = 3)}$$

Also you otherwise write down $\frac{1}{4}\frac{1}{3}$ {*that is* $\frac{1}{4} + \frac{1}{3}$} *in this way; and you multiply the* 1 $\{n_{21} = 1\}$ *which is over the* 3 $\{d_{21} = 3\}$ *by the* 4 $\{d_{11} = 4\}$; *there will be* 4 *that you write above the* $\frac{1}{3}$ {*as* $(n_{22} = 1) \times (d_{12} = 4) = 4$} *and the* 1 $\{n_{11} = 1\}$ *which is over the* 4 $\{d_{11} = 4\}$, *you multiply by the* 3 $\{d_{21} = 3\}$; *there will be* 3 *that you write above the* $\frac{1}{4}$ {*as* $(n_{12} = 1) \times (d_{22} = 3) = 3$} *and you add them together; there will be* 7 {*as* $(n_{22} = 1) \times (d_{12} = 4) + (n_{12} = 1) \times (d_{22} = 3) = 7$} *that you divide by the* 3 $\{d_{21} = 3\}$ *and the* 4 $\{d_{11} = 4\}$ *that are under the fractions, that is by* 12 {*calculated as* $(d_{13} = 4) \times (d_{23} = 3) = 12$}; *the quotient similarly will be* $\frac{7}{12}$ *for the addition; and you know this is to add* $\frac{1}{3}$ *and* $\frac{1}{4}$ *which is* $\frac{1}{4}\frac{1}{3}$ {*that is* $\frac{1}{4} + \frac{1}{3}$} *that are parts of the unity; they are indeed* $\frac{7}{12}$ *of the unity; and thus you understand for the addition of all fractions.*

Rearranging terms gives the usual algebraic form of the addition of fractions, which is the same as our algebraic transliteration of Brahmegupta's Addition A in the Ganita, Verse 2, [4], [9]. However, Brahmegupta's addition also explicitly applies to all negative numbers and zero.

Definition 2.24. *For all non-negative decimal integers n_1, d_1, n_2, d_2 the Addition A sum, $+$, of fracterms $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is given by:*

$$\frac{n_1}{d_1} + \frac{n_2}{d_2} = \frac{n_1 d_2 + n_2 d_1}{d_1 d_2}$$

Thus, Fibonacci, 1202 CE, uses Brahmegupta's, 628 CE, Addition A, not Brahmegupta's and Bhascara's, 1150 CE, Addition B [4], [9]. Both forms of addition sum finite fractions and fractions with a zero denominator, but Addition A is more laborious when adding fractions with a common denominator of any kind and it is more absorptive when adding fractions with a common denominator of zero.

Continuing on page 100, Fibonacci combines the account of special and general addition to describe subtraction. We do not add parenthetic comments because this would impose our disambiguation of Fibonacci's conflation of the special and general accounts. We leave the reader to form an opinion of the text.

And if you will wish to subtract $\frac{1}{4}$ from $\frac{1}{3}$ then the 3 that is written above the $\frac{1}{4}$ that is a quarter of 12, you subtract from the 4 that is written above the $\frac{1}{3}$ that is a third of 12; there will remain 1 which you divide by the found 12, or by the 3 and the 4 that are under the fractions; the difference for the said subtraction will be $\frac{1}{12}$, that is $\frac{1\ 0}{2\ 6}$.

Here the compound fraction is $\frac{1\ 0}{2\ 6} = \frac{1}{2\times 6} + \frac{0}{6} = \frac{1}{12}$. Notice that the fraction $\frac{0}{6} = 0$ supports our contention that Fibonacci can write zero as a fraction, see Definition 2.14, and can add zeros, see Definition 2.6. There are multiple examples of these operations on zero in Fibonacci's text.

Definition 2.25. *For all non-negative decimal integers n_1, d_1, n_2, d_2, such that $\frac{n_1}{d_1} > \frac{n_2}{d_2}$ the Addition A difference, $-$, of fracterms $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is given by:*

$$\frac{n_1}{d_1} - \frac{n_2}{d_2} = \frac{n_1 d_2 - n_2 d_1}{d_1 d_2}$$

Notice that Fibonacci's subtraction of fractions is partial. Not only must he have the left argument, $\frac{n_1}{d_1}$, greater than the right argument, $\frac{n_2}{d_2}$, but he also cannot subtract any fractions with a zero denominator because, in this circumstance, he cannot resolve the greater than constraint.

By contrast, Brahmegupta's, 628 CE, subtraction has the same algebraic form but applies to all positive, zero, and negative numbers, without side conditions. See the Ganita, Verse 2, [4], [9].

Continuing on page 100, Fibonacci describes the division of fractions. We begin with his special algorithm:

$$\frac{(n_{11} = 1)}{(d_{11} = 3)} \div \frac{(n_{21} = 1)}{(d_{21} = 4)} = \frac{(d_{22} = 4)}{(d_{21} = 3)}$$

And if you will wish to divide $\frac{1}{3}$ by $\frac{1}{4}$ then you divide by the 3 $\{d_{11} = 3\}$ the 4 $\{d_{21} = 4\}$ that is above the $\frac{1}{3}$ and you will have $\frac{1}{3}1$ for the fraction and integer $\{\frac{1}{3}1 = \frac{4}{3}\}$.

Fibonacci then presents a Euclidean proof of the correctness of this division, using Euclid's account of proportions:

The ratio of $\frac{1}{3}$ to $\frac{1}{4}$ for example, is as the ratio of twelvefold $\frac{1}{3}$ to twelvefold $\frac{1}{4}$ that is as 4 is to 3, so is $\frac{1}{3}$ to $\frac{1}{4}$. Therefore the division of $\frac{1}{3}$ by $\frac{1}{4}$ results in the same which results from the division of 4 by 3;

Fibonacci's proof, just presented, relies on Euclid's proportionality of ratios and exploits Fibonacci's identification of ratios, $x : y$, with division, $x \div y$, expressed as fractions, $\frac{x}{y}$. We now present our argument, of the same form, which proves that division by zero exists. Following Fibonacci, $x : y = x \div y = \frac{x}{y}$. But, if the ratio $x : y$ exists, then so does the reciprocal ratio $y : x$. Now $0 : x = 0 \div x = \frac{0}{x}$ exists by Definition 2.14. Therefore, the reciprocal $x : 0 = x \div 0 = \frac{x}{0}$ exists. Thus, we have proved the following corollary.

Corollary 2.26. *Division by zero, $\frac{x}{0}$, exists because multiplication by zero exists in the reciprocal fraction, $\frac{0}{x}$.*

But this is to say that the Euclidean mathematics, of about 300 BCE, supported multiplication and division by zero as soon as zero became available in 628 CE!

It was previously noted by Fowler [10] that, prior to Euclid, the Greek algorithm of Anthyphairesis supported division by zero, even though zero was not known to the Greeks at that time. Thus, we have a second reason to believe that Ancient Greek mathematics provided a foundation for the later development of division by zero.

On page 101, Fibonacci describes a general algorithm for the division of fractions.

$$\frac{(n_{11} = 2)}{(d_{11} = 3)} \div \frac{(n_{21} = 4)}{(d_{21} = 5)} = \frac{(n_{12} = 2) \times (d_{22} = 5)}{(n_{22} = 4) \times (d_{12} = 3)}$$

And if you wish to divide $\frac{2}{3}$ by $\frac{4}{5}$ then you divide the 10 $\{(n_{12} = 2) \times (d_{22} = 5) = 10\}$ *by the* 12 $\{(n_{22} = 4) \times (d_{12} = 3) = 12\}$*; the quotient will be $\frac{5}{6}$.*

After many such examples, we adopt the following definition.

Definition 2.27. *For all non-negative decimal integers n_1, d_1, n_2, d_2 the division, $\div$, of fracterms $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is given by:*

$$\frac{n_1}{d_1} \div \frac{n_2}{d_2} = \frac{n_1 \times d_2}{n_2 \times d_1}$$

With a slight rearrangement of terms, This is the same as our transliteration of Brahmegupta's division in the Ganita, Verse 4, [4], [9]. However, Brahmegupta's division also explicitly applies to all negative numbers and zero numerators and denominators.

All of Fibonacci's fractional arithmetic is consistent when we specify the numerator and denominator of every fraction because the operations of fractional arithmetic are carried out in decimal integer arithmetic, which is a subset of real arithmetic, and real arithmetic is consistent.

Now we have fractional addition, we can demonstrate a further inconsistency with Fibonacci allowing, but not claiming, $\frac{0}{0} = 0$.

Consider the sum $x = \frac{2}{3} + 0$. There are two ways to calculate this sum, which have different answers.

Firstly, agreeing with rational arithmetic:

$$\begin{aligned} x &= \frac{2}{3} + 0 \\ &= \frac{2}{3} + \frac{0}{1} \text{ by Definition 2.14} \\ &= \frac{2 \times 1 + 0 \times 3}{3 \times 1} \text{ by Definition 2.24} \\ &= \frac{2}{3} \end{aligned}$$

Secondly, departing from rational arithmetic:

$$\begin{aligned} x &= \frac{2}{3} + 0 \\ &= \frac{2}{3} + \frac{0}{0} \text{ by Theorem 2.17} \\ &= \frac{2 \times 0 + 0 \times 3}{3 \times 0} \text{ by Definition 2.24} \\ &= \frac{0}{0} \\ &= 0 \text{ by Theorem 2.17} \end{aligned}$$

But $\frac{2}{3} \neq 0$.

The equality $\frac{0}{0} = 0$ causes inconsistency in Brahmegupta's 628 CE arithmetic [4] and causes inconsistency in Fibonacci's arithmetic. However, three ways to dissolve this inconsistency are known. Firstly, one can adopt the Suppes-Ono Convention $\frac{x}{0} = 0$, see [3], [12], but disallow the substitution of fractions $x/0$ and the number 0 in fractional arithmetic. Secondly, one can adopt bottom, $\perp$, as an absorptive element that is used as an error flag in the context $0/0 = \perp$. Bottom is used, for example, in the algebraic structure of meadows, [5], [6], [7], [8]. Thirdly, one can adopt nullity, $\Phi = \frac{0}{0}$, as an algebraic number, different from zero and bottom, in the transreal numbers [1], [2].

2.8 Chapter 8

Chapter 8 starts on page 127. It deals with proportions of price and quantity, where the aim is to discover an unknown quantity from stated proportions. Even a cursory reading of this chapter and later ones discovers the goods and concerns of 13th Century merchants.

2.9 Chapter 9

Chapter 9 starts on page 179. It deals with the use of proportions to obtain the fair barter of goods when the prices of goods are known. It also deals with the feeding of horses when the daily consumption of food per horse is known.

2.10 Chapter 10

Chapter 10 starts on page 213. It deals with the use of proportions to arrive at fair shares of profit and loss for merchants engaged in a company.

2.11 Chapter 11

Chapter 11 begins on page 227. It deals with the use of linear equations and linear inequalities to alloy precious metals and to mix goods.

2.12 Chapter 12

Chapter 12 begins on page 259. It deals with a miscellaneous collection of techniques but is mainly concerned with the sum of series and linear equations. Fibonacci's famous problem about breeding rabbits starts on page 439.

A negative subtotal is demonstrated at (15,322,626), despite the absence of an explicit account of negative numbers.

2.13 Chapter 13

Chapter 13 starts on page 447. It solves general linear equations by the method of double false position, known as Elchataym.

Solving general linear equations requires the multiplication and addition of positive, negative, and zero numbers. But it seems to us that Fibonacci calculates with negative numbers by treating them as positive magnitudes that are taken in the direction opposite to positive numbers. The eventual solution to his problems is always a sum of the form $z-y > 0$.

Thus, Fibonacci demonstrates an implicit understanding of zero and negative numbers that he does not make explicit anywhere in Liber Abaci.

2.14 Chapter 14

Chapter 14 starts on page 489. It gives methods for computing square and cube roots. Thus, Fibonacci's fractional arithmetic is extended to these computable irrational numbers. Similar techniques were known to Brahmegupta, 628 CE, and Bhascara, 1150 CE [9].

2.15 Chapter 15

Chapter 15 begins on page 531. It deals with quadratic equations.

2.16 Discussion

In previous work [4] we categorised the Hindu arithmetics as being total and paraconsistent or consistent. This categorisation also applies to Fibonacci.

The Hindu mathematicians had algorithms for calculating with decimal integers. These algorithms had been tested in both natural observation in astronomy and geography, and in social settings such as trade. The Hindu decimal integers passed from India, via Arabia, to Fibonacci. Eventually, they became part of the real number system, which is consistent. Wherever calculations with the decimal integers produce a decimal integer as result, real arithmetic produces the same integer result, so the decimal integers are self-consistent, albeit there are slight differences in the algebraic structure of the decimal integers used by each author. One way to characterise this state of affairs is to say that the decimal integer arithmetics are *deterministic,* which is enough to make an arithmetic useful, regardless of its other algebraic properties.

Brahmegupta's 628 CE description of addition is ambiguous. It has two forms of addition, which we call Addition A and Addition B. Bhascara, 1150 CE, used Addition B and Fibonacci, 1202 CE, used Addition A. We do not known which form of addition Brahmegupta used in calculations, though this might be known to historians. Each author describes the remaining three fundamental operations of subtraction, multiplication, and division in the same way, though the algebraic properties of the lexical operations differ between authors.

Brahmegupta describes trichotomous numbers that are negative, zero, or positive; but he does not impose trichotomy and, indeed, he has some non-trichotomous numbers. His arithmetic is deterministic, but his description of its properties is inconsistent, because he asserts $\frac{0}{0} = 0$. We say that his deterministic but inconsistent arithmetic is *paraconsistent.*

Bhascara accepted most of Brahmegupta's work, but dropped the claim $\frac{0}{0} = 0$, leaving $\frac{0}{0}$ as an unanalysed tetrachotomous number. This emendation made Bhascara's description consistent with his arithmetic. Bhascara also identified the fractions with a non-zero numerator and a zero denominator as being unsigned infinities, though his theological justification for this is opaque. See the Vija-Ganita, Verse 16 [4], [9].

Fibonacci recognises positive numbers so he has at least a unichotomy of numbers. He does not seem to regard zero as a number, but he correctly multiplies and adds zeros, so we are inclined to say he has at least a dichotomy of objects that take part in his arithmetic.

Fibonacci speaks of *debts* and *profits.* He can calculate the magnitude of a loss and can sum a greater profit to a smaller loss to produce a profit. Thus, Fibonacci has some handling of the addition of positive and negative numbers, so we say he has a trichotomy of objects that take part in his arithmetic. But Fibonacci has lost the explicit idea of classifying objects by sign.

Fibonacci gives multiple definitions of division. His statement of division as a fraction, $n \div d = \frac{n}{d}$, is total; but his explanation of division in terms of a multiplicative inverse is inconsistent; but if the numerator and denominator of a fraction are always given as decimal integers or as computable irrational numbers, then his division is deterministic, which is to say it is total and paraconsistent. However, a slight re-write of Fibonacci's description of division produces division by the modern multiplicative inverse, which is partial and consistent. A further slight re-write gives a total and consistent definition of division via the modern multiplicative inverse, where this inverse exists, otherwise the quotient is an irreducible fraction of the form $\frac{z}{0}$, for all non-negative z. Thus, we see that there is a spectrum of ways to totalise Fibonacci's arithmetic and make it consistent.

Fibonacci gave a Euclidean proof of the correctness of the division of fractions, but he did not notice the corollary of his proof that division by zero, $\frac{x}{0}$, exists because multiplication by zero exists in the lexically reciprocal fraction, $\frac{0}{x}$. Thus, the Euclidean mathematics, of about 300 BCE, supported multiplication and division by zero as soon as zero became available in 628 CE!

In Summary, Fibonacci obtained the decimal integers, fractions, and some computable irrational numbers from India, via Arabia, but he lost some algebraic structure. Fibonacci lost: tetrachotomy, explicit negative numbers, the additive inverse, the totality of subtraction, sign conventions, infinities, and division by zero.

Despite claiming totality of division, Fibonacci gives no example of division by zero and does not discuss this possibility anywhere in Liber Abaci. This is the strongest evidence that Fibonacci, 1202 CE, did not have access to Brahmegupta's 628 CE or Bhascara's 1150 CE account of division by zero.

Furthermore, if we read Fibonacci as allowing, but not claiming, division by zero, then his division of decimal integers by zero is inconsistent and his division of fractions by zero is paraconsistent – both involve inconsistency of some kind, which was an obstacle before the mid-twentieth century, and could not reasonably have passed into 13th century practice.

2.17 Conclusion

Fibonacci learned Hindu arithmetic and Hindu-Arabic algebra from Muslim North African Arabs and published a Latin treatise on it, incorporating his own mathematical developments. However, he lost: tetrachotomy, explicit negative numbers, the additive inverse, the totality of subtraction, sign conventions, infinities, and division by zero. Arguably, he also lost zero as an algebraic number, having access to zero only as a placeholder.

Fibonacci gives no example of division by zero and does not discuss the possibility of division by zero anywhere in Liber Abaci, which is the strongest evidence that he did not have access to Hindu division by zero.

However, Fibonacci did introduce a partial inconsistent multiplicative inverse, which prefigured the modern partial consistent multiplicative inverse, which can, in turn, be totalised consistently by taking $x/0$ as an irreducible fraction, in the same way as Bhascara did in 1150 CE.

Following the failure to transmit Brahmegupta's 628 CE division by zero and Bhascara's 1150 CE division by zero to Fibonacci, Fibonacci failed to define division by zero in a consistent way, thereby contributing to the loss of division by zero.

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