

Al-Khwarizmi and the Loss of Division by Zero

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Submitted: 24 August 2026

Revised: 7 October 2026

Abstract

We review two literal English translations, with pragmatic notes, of surviving accounts of Muhammad ibn Musa al-Khwarizmi's Arabic writings on the Hindu decimal number system and on algebra. Al-Khwarizmi, about 780-850 CE, showed no knowledge of division by zero as described in Sanskrit by Brahmagupta, 628 CE. Indeed, al-Khwarizmi's definition of division is inconsistent with division by zero. Al-Khwarizmi has lost Brahmagupta's: fractions as a syntactic object n/d , negative numbers, trichotomy, tetrachotomy, and division by zero, though he was close to obtaining general negative numbers from his sign convention for variables. Thus al-Khwarizmi contributed to the loss of division by zero. We hypothesise that Brahmagupta's division by zero did not diffuse from Sanskrit until Colebrooke's English translation of 1817 CE.

1 Introduction

Previously, we argued that a paraconsistent total arithmetic, including division by zero, was published in Sanskrit by Brahmagupta in 628 or 629 CE [1]. This arithmetic was maintained, made consistent, and further developed in Sanskrit for a thousand years, up to at least 1621 CE [1], [3]. Thereafter, English gradually replaced Sanskrit as a regional lingua franca of science, bringing to an end the development of science in Sanskrit.

We also argued that when Fibonacci promoted Hindu-Arabic arithmetic from 1202 CE [7], he popularised a partial Sanskrit arithmetic without division by zero [2]. Thus, we argue, Fibonacci contributed to the loss of division by zero. In our paper, we wondered if the loss of division by zero occurred earlier in Arabia? We now argue that this is the case, specifically that Muhammad ibn Musa al-Khwarizmi, about 780-850 CE, promoted partial Hindu-Arabic arithmetic and algebra, without division by zero. Indeed, as we shall show, al-Khwarizmi's definition of division is inconsistent with division by zero. Hence, we conclude that al-Khwarizmi contributed to the loss of division by zero.

There are no surviving Arabic copies of al-Khwarizmi's treatise on arithmetic, but there are a handful of Latin manuscripts that cite al-Khwarizmi and set out what the Latin authors understood to be al-Khwarizmi's arithmetic. Crossley and Henry [4] provide a literal English translation of the earliest such manuscript and give a pragmatic account of al-Khwarizmi's calculations in their appendix. However, the translated Latin manuscript is thought to be an imperfect copy of an earlier Latin manuscript that has now been lost. Thus, nothing is known about the provenance of the earlier text and how it related to al-Khwarizmi's original. We must therefore be cautious in ascribing mathematical knowledge or the lack of mathematical knowledge to al-Khwarizmi himself.

Today, there is one partial Arabic copy of al-Khwarizmi's writing on algebra, the *Kitab al-Jabr w'al muqabala*, written around 825 CE. Rosen, 1831 CE, [6] gives a literal English translation of it, together with pragmatic footnotes written in modern mathematical notation by an anonymous co-author, see [6] pages xv to xvi. Rosen includes what we take to be an Arabic copy of the manuscript. He bemoans the lack of diacritical marks in the Arabic, which makes some of the mathematical content obscure. However, we can be reasonably certain that this text does represent al-Khwarizmi's mathematical knowledge, if only imperfectly.

Rosen's introduction to his translation [6] of the *al-Jabr*, describes the historical context of al-Khwarizmi's work on algebra. We have augmented his account by reading the on-line entries of the main actors in the *Encyclopedia Britannica* [5].

Brahmagupta's work was conveyed to the caliphate by an anonymous Indian astronomer about 773 CE. The Indian document has now been lost but, before it was lost, it was translated into Arabic by Mohammed Ben Ibrahim al-Fazari.

In its turn, al-Fazari's translation has now been lost but, before it was lost, al-Mamun requested that al-Khwarizmi abridge al-Fazari's Arabic translation of Brahmagupta's work. Later, as caliph, from about 813 CE to about 833 CE, al-Mamun commanded al-Khwarizmi to write a popular

work on algebra, which is the al-Jabr translated by Rosen. Thus, there is direct evidence of Brahmegeupta's influence on al-Khwarizmi.

Later, Hieronymus Cardanus, also known as Gerolamo Cardano, 1501-1576 CE, credited Fibonacci with the view that al-Khwarizmi invented algebra. We reviewed Fibonacci's arithmetic [7] in [1], where we argued that Fibonacci lost Brahmegeupta's division by zero. Thus, there is indirect evidence that al-Khwarizmi influenced Fibonacci and there is also direct evidence in that Fibonacci cited al-Khwarizmi as Maumhet, see [7]. Rosen cites several sources that say al-Khwarizmi did not invent algebra, but was the first Mohammedan to write about the explicit solution of equations. Today, Al-Khwarizmi is widely credited with influencing Western mathematics. Indeed his name gave rise to the word *algorithm* and the title of his al-Jabr gave us the word *algebra*.

Thus, there is clear evidence that Brahmegeupta did have an influence on al-Khwarizmi who, in his turn, influenced Western mathematics; but where and when was Brahmegeupta's division by zero lost? Was it lost by the anonymous Indian astronomer who brought Brahmegeupta's work to Arabia? Was it lost in Mohammed Ben Ibrahim al-Fazari's Arabic translation? Was it lost in al-Khwarizmi's abridgement? These are questions we cannot answer, because the necessary documents have been lost. Instead, we demonstrate that al-Khwarizmi could not divide by zero, thereby making the loss certain.

Rosen [6] cites Colebrooke's English translation [3] of Brahmegeupta's arithmetic, but why did Rosen fail to remark on division by zero and its loss in al-Khwarizmi's text?

Al-Khwarizmi's treatise on arithmetic refers to his earlier treatise on algebra, Crossley and Henry [4] Page 110, with their parenthetical comment: *And already I have revealed in the book of algebra and almuqabalah, i.e., restoration and opposition [Rosen 1831], that every number is composite and that every number is put together above one.*

We present these texts in the reverse of their historical order because we need al-Khwarizmi's arithmetic to establish that he cannot divide by zero. Thereafter we obtain more of the algebraic structure of his arithmetic from his earlier treatise on algebra.

2 Arithmetic

In this section, all page numbers refer to Crossley and Henry [4], where the word *circle* means the zero numeral, written as *o* in the Latin manuscript. We write quotations from the translation in *italics* and also use italics for emphasis. The different uses of italics should be clear from context.

Al-Khwarizmi does not have negative numbers, so when we transcribe al-Khwarizmi's arithmetic into modern notation, we use non-negative numbers, specifically non-negative integers. Thus, al-Khwarizmi has lost Brahmegeupta's negative numbers as described in [1], Section 2.2, beginning on Page 12, and the Cuttaca, Verses 31-36, in [3].

Al-Khwarizmi presents algorithms for: the addition of non-negative integers, the subtraction of non-negative integers with a non-negative result, the multiplication of non-negative integers, and the long division of a positive integer by a positive integer, such that the divisor is smaller than the dividend, giving a positive quotient greater than unity. The long division algorithm is so vague that we cannot analyse long division of zero or long division by zero. Al-Khwarizmi also describes how to check calculations by casting out nines.

Al-Khwarizmi presents an arithmetic of *fractions*, but these are not numbers of the form n/d , instead they are calculations of hours, minutes, and seconds in base sixty. In al-Khwarizmi's view, minutes are sixtieth *fractions* of an hour and seconds are sixtieth *fractions* of a minute. In other words, al-Khwarizmi's *fractions* are *dimensional units* that are calculated as integers. Al-Khwarizmi can calculate *proportions*, but this is done by scaling up by a common measure, performing integer arithmetic, and then scaling down. Thus, al-Khwarizmi has lost Brahmagupta's fractions of the form n/d as described in the whole of [1], and Verses 2-4 of the Ganita and Verses 31-36 of the Cuttaca in [3].

Let us state the obvious. A *placeholder zero* marks an empty place in the decimal place-value notation, but if any two numbers can be added then the algebraic properties of addition are known, and if they can be subtracted then the algebraic properties of subtraction are known, and if they can be multiplied then the algebraic properties of multiplication are known, and if they can be divided then the algebraic properties of division are known. Thus, a placeholder zero is at the minimal extreme of a spectrum of algebraic structure; but as a number is evaluated by adding the values in its places, a placeholder zero always has some algebraic structure – that of addition! Thus, every zero is an *algebraic zero*, whence there is very little value in calling an algebraic zero a *placeholder zero*. Henceforth, we do not use the term *placeholder zero*.

Al-Khwarizmi does not exclude division by zero, so let us now consider the consequences of allowing it.

Page 116-117: *Because every circle that is multiplied by some number is nothing, that is to say no number arises from it and whatever is multiplied by a circle is likewise nothing.*

We transcribe this total definition of commutative multiplication by zero into modern notation as follows.

Definition 2.1. *For all non-negative integers, z , it is the case that $0 \times z = z \times 0 = 0$.*

Pages 122-123, with our parenthetic comments in Latin braces {like this}: *And know that as regards every number $\{n\}$ that is divided by another number $\{d\}$, if what is extracted $\{q\}$ from that which is divided {as $q = n/d$ } is multiplied by that by which it is being divided $\{d\}$, the first number $\{n\}$ will return, i.e., the number [that] is being divided {as*

$n = q \times d = (n/d) \times d$. Of this an example is: that when you divide L $\{n = 50\}$ by X $\{d = 10\}$, there will result what is owed to one, i.e., five $\{q = n/d = 50/10 = 5\}$. And when you multiply that which has resulted to you from the division, i.e., five $\{q = 5\}$, by that by which you are dividing, which is X $\{d = 10\}$, there will return the first number, i.e., L $\{n = q \times d = (n/d) \times d = (50/10) \times 10 = 50\}$.

We transcribe this total definition of division into modern notation as follows.

Definition 2.2. Let n, d, q be non-negative integers, then $n \div d = q \iff n = d \times q$.

Theorem 2.3. Al-Khwarizmi's division, as given in Definition 2.2, is inconsistent.

Proof.

$$\begin{aligned} 1 \div 0 = q &\iff 1 = 0 \times q \text{ by Definition 2.2} \\ &= 0 \text{ by Definition 2.1} \end{aligned}$$

□

Whence al-Khwarizmi cannot divide by zero.

3 Algebra

In this section, all page numbers refer to Rosen [6]. We write quotations from Rosen's translation in *italics* and present the footnotes supplied by Rosen's anonymous co-author, see [6] pages xv to xvi, in upright text, though we write the mathematical equations in the footnotes in a mixture of *italic* and upright text, following the usual conventions.

We begin by reaffirming that al-Khwarizmi cannot divide by zero. We then obtain some of the algebraic structure of al-Khwarizmi's arithmetic. A subtle point is that al-Khwarizmi can describe a negative variable, $-a$, in a subtraction, $x - a$, but this subtraction can have only a non-negative result, $x - a \geq 0$, which is further tightened to $x - a > 0$ in his examples. We gloss this by saying that al-Khwarizmi can have context-sensitive negative variables, but cannot generally have negative numbers. The subtlety is that negative variables occur only in the context $x - a > 0$.

Page 46, with our parenthetic comments in Latin braces {like this}: *Observe that, in every case, where any two quantities whatsoever are divided, the first $\{a\}$ by the second $\{b\}$ and the second $\{b\}$ by the first $\{a\}$, if you multiply the quotient of the one division $\{a/b\}$ by that of the other $\{b/a\}$, the product is always one.*

Footnote:

$$\frac{a}{b} \times \frac{b}{a} = 1$$

But this is contradictory when a or b is zero, as shown in Theorem 2.3, whence, again, al-Khwarizmi cannot divide by zero.

Notice, too, that al-Khwarizmi's claim is total *any [...] quantities whatsoever*, but this claim does not apply to zero, therefore, we conclude that al-Khwarizmi does not regard zero as a quantity or number. This distinction between zero and numbers pervades al-Khwarizmi's text. We gloss this by saying that al-Khwarizmi has a dichotomy of positive numbers in one class and zero in another class, whence al-Khwarizmi has lost Brahmagupta's trichotomy and tetrachotomy. See Pages 3 and 5, and Section 2.2 in [1], and Verses 31-36 of the Cuttaca in [3].

Page 21: *Whenever one number is to be multiplied by another, the one must be repeated as many times as the other contains units.*

Footnote: If x is to be multiplied by y , x is to be repeated as many times as there are units in y .

Page 21-22: *If there are greater numbers combined with units to be added to or subtracted from them, then four multiplications are necessary; namely, the greater numbers by the greater numbers, the greater numbers by the units, the units by the greater numbers, and the units by the units.*

Footnote: If $x \pm a$ is to be multiplied by $y \pm b$, x is to be multiplied by y , x is to be multiplied by b , a is to be multiplied by y , a is to be multiplied by b .

Notice that the algebra in the footnote is an account of the modern form of distributivity, but Rosen's anonymous co-author, see [6] pages xv to xvi, has missed out the side condition *If there are greater numbers combined with units to be added to or subtracted from them*. This side condition is $x > a > 0$ and $y > b > 0$. The side condition $a, b > 0$ occurs because zero is not a unit! Thus, we have the curious position that the algebra implied by al-Khwarizmi's verbal description is more general than his arithmetic. This becomes more striking in the subsequent paragraphs of al-Khwarizmi's text, as we now see.

Page 22: *If the units, combined with the greater numbers, are positive, then the last multiplication is positive; if they are both negative, then the fourth multiplication is likewise positive. But if one of them is positive, and one negative, then the fourth multiplication is negative.*

Footnote with our parenthetical comment: In multiplying $(x \pm a)$ by $(y \pm b)$ {we have:}

$$+a \times +b = +ab$$

$$-a \times -b = +ab$$

$$+a \times -b = -ab$$

$$-a \times +b = -ab$$

This sign convention appears in Brahmagupta, Section 2.2 in [1], and Verses 31-36 of the *Cuttaca* in [3], but there it applies to all numbers, whereas in al-Khwarizmi it applies to only those numbers that satisfy the side condition $x > a > 0$ and $y > b > 0$, which is to say that Brahmagupta has negative numbers, while al-Khwarizmi does not.

Page 22: *For instance, “ten and one to be multiplied by ten and two.” Ten times ten is a hundred; once ten is ten positive; twice ten is twenty positive, and once two is two positive; this altogether makes a hundred and thirty-two.*

We faithfully present the next three footnotes in modern notation. The interested reader may consult Rosen [6] for the original layout.

Footnote:

$$\begin{aligned}(10 + 1) \times (10 + 2) &= (10 \times 10) + (1 \times 10) + (2 \times 10) + (1 \times 2) \\ &= 100 + 10 + 20 + 2 \\ &= 132\end{aligned}$$

Page 22-23: *But if the instance is ten less one, to be multiplied by ten less one,” then ten times ten is a hundred; the negative one by ten is ten negative; the other negative one by ten is likewise ten negative, so that it becomes eighty: but the negative one by the negative one is one positive, and this makes the result eighty-one.*

Footnote:

$$\begin{aligned}(10 - 1)(10 - 1) &= (10 \times 10) + (-1 \times 10) + (-1 \times 10) + (-1 \times -1) \\ &= 100 - 10 - 10 + 2 \\ &= 82\end{aligned}$$

Page 23: *Or if the instance be “ten and two, to be multiplied by ten less one,” then ten times ten is a hundred, and the negative one by ten is ten negative; the positive two by ten is twenty positive; this together is a hundred and ten; the positive two by the negative one gives two negative. This makes the product a hundred and eight.*

$$\begin{aligned}(10 + 2) \times (10 - 1) &= (10 \times 10) + (-1 \times 10) + (+10 \times 2) + (-1 \times 2) \\ &= 100 - 10 + 20 - 2 \\ &= 108\end{aligned}$$

Thus, al-Khwarizmi explains the distributivity of positive and negative variables, but only in contexts where the eventual sum is positive. We gloss this by saying that al-Khwarizmi can have context-sensitive negative variables but cannot have negative numbers. The subtlety is that negative variables occur only in the context $x - a > 0$.

At this point, al-Khwarizmi is very close to having negative numbers. He has a total sign convention of variables, but can apply it to numbers only in the context $x - a > 0$. Had he applied the sign convention of variables to all numbers, he would have had both positive and negative numbers, together with zero as given in Definition 2.1. It is interesting that in the Arabic language, al-Khwarizmi obtained a sign convention of variables *before* a sign convention of numbers!

Al-Khwarizmi goes on to describe how to obtain positive solutions of a quadratic equation, how to solve some geometrical problems, how to solve equations of one and two variables, and how to distribute money in fixed proportions. Proportions are calculated by scaling up by a common measure, performing integer arithmetic, and then scaling down. There is no indication, in al-Khwarizmi’s text, that he used syntactic fractions of the form n/d as Brahmeḡupta did.

3.1 Conclusion

Al-Khwarizmi popularised the Indian decimal place-value arithmetic in Arabic, whence it diffused via Latin to the Western world, but he lost Brahmeḡupta’s: fractions as a syntactic object n/d , negative numbers, trichotomy, tetrachotomy, and division by zero, though he was close to obtaining general negative numbers from his sign convention for variables.

It is not clear where or when Brahmeḡupta’s division by zero was first lost to Arabia. Nor do the documents survive that would let us answer these questions. All that remains for us is to hypothesise that Brahmeḡupta’s division by zero did not diffuse from Sanskrit until Colebrooke’s English translation of 1817 CE.

References

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